Total Domination Number of Butterfly Graph

Indrani Kelkar\(^1\), B. Maheswari\(^2\)
\(^1\)Department of Mathematics, Acharya Institute of Technology, Bangalore, INDIA
\(^2\)Department of Applied Mathematics, Sri Padmavati Mahila Visvavidyalayam, Tirupati, India.

ABSTRACT
Butterfly graphs are very important structures in computer architecture and communication techniques. Total Domination is an important parameter as it ensures connectivity in the network even after failure of few of the communication points. In this paper we find total domination number of BF\((n)\) and deduce a relation between the domination number and total domination number of BF\((n)\) as

\[
\gamma (\text{BF}(2)) = 2^2 = \gamma (\text{BF}(2)) + 2
\]

\[
\gamma (\text{BF}(3)) = 2^3 = \gamma (\text{BF}(3)) + 2
\]

\[
\gamma (\text{BF}(n)) = \gamma (\text{BF}(n)) \quad \text{for } n > 3
\]

Keywords— Butterfly graph, domination number, total domination number

I. INTRODUCTION
Berge [3] presents the problem of 5 queens as - Can we place 5 queens on a chess board such that every square on the board is covered by at least one queen? Another problem is that not only one or more queens dominate the squares, but each queen is dominated by another queen. The solution of this problem introduces the concept of total dominating sets in graphs.

Total Dominating sets were introduced by Cockayne, Dawes and Hedetniemi [4]. Some results on total domination can also be seen in Allan, Laskar and Hedetniemi [1]. In this paper we present the results on total domination number of butterfly graph BF\((n)\).

Butterfly Graph BF\((n)\) : The vertex set V of BF\((n)\) is the set of ordered pairs \((\alpha; \nu)\) where \(\alpha \in \{0, 1, 2, \ldots, n-1\}\) and \(\nu = x_{n-1} x_{n-2} \ldots x_1 x_0\) is a binary string of length \(n\) where \(x_i = 0\) or 1. There is an edge from a vertex \((\alpha; \nu)\) to a vertex \((\alpha'; \nu')\) where \(\alpha' = \alpha + 1 \, (\mod n)\) and \(x_j = x'_j \quad \forall \ j \neq \alpha'.\) A butterfly graph BF\((n)\) is an n - partite graph with \(n\) levels. Each level \(L_k\) for \(k = 0, 1, \ldots, n-1\) has \(2^n\) vertices and \(L_k = \{ (k; \nu) / \nu = x_{n-1} x_{n-2} \ldots x_1 x_0, \ x_i = 0\ \text{or} \ 1\}.\) Using decimal representation of the binary string we can write \(L_k = \{ (k; m) / \text{where } k = 0, 1, 2, \ldots, n-1 \text{and } m = \sum x_j 2^j, \ j = 0,1, \ldots, n-1 \}\).

II. SPECIAL RECURSIVE CONSTRUCTION OF BF(4k)
The recursive construction given by Barth and Raspaud [2] constructs BF\((n)\) from 2 copies of BF(n-1), which gives that first 4 consecutive levels \(L_0, L_1, L_2, L_3\) consist of \(2^{n-1}\) copies of BF\((4)\) in BF\((n)\) for \(n > 4\). Now we present a special recursive construction of BF\((n)\) for \(n = 4k\) from BF\((4)\), which is of great advantage for finding dominating sets of BF\((4k)\) from the dominating set of BF\((4)\).

Consider \(2^4\) copies of the graph BF\((4(k-1))\) in levels \(L_0, L_1, L_2, \ldots, L_{4k-5}\) and place them next to each other. Now call a set of \(2^{n-3}\) vertices in each copy in these levels as a vertex group. Define a set of parallel edges between these vertices as an edge between vertex groups. Now the 4 levels \(L_{4k-4}, L_{4k-3}, L_{4k-2}\) and \(L_{4k-1}\) consist of \(2^4\) vertex groups in each level with edges as defined in BF\((4)\) without winged edges. We shall call this a pattern of BF\((4)\), without winged edges. Thus BF\((4k)\) can be obtained recursively from BF\((4(k-1))\) by taking \(2^4\) copies of BF\((4(k-1))\), adding 4 levels of \(2^4\) vertex groups and defining the edges between these vertex groups in the same way as the edges between vertices of BF\((4)\). Figure 2 illustrates this process for \(k = 2\), namely for BF\((8)\).

![Figure 1: Special recursive construction of BF(8)](Image)

Indrani [6] has given minimal dominating sets of butterfly graphs and the domination numbers of BF\((n)\) found are as follows.
\( \gamma(BF(n)) = 2 \) if \( n = 2 \)
\( = 6 \) if \( n = 3 \)
\( = (2k+r) 2^{n-1} \) if \( n = 4k+r, r = 0,1, \) and \( k \in Z^+ \)
\( = (k+1) 2^n \) if \( n = 4k+r, r = 2,3, \) \( k \in Z^+ \).

Because of the \( n \)-partite structure of \( BF(n) \), we choose vertices into the total dominating set from two consecutive levels say \( L_i \) and \( L_{i+1} \) which together dominate all the vertices of 4 levels \( L_{i-1}, L_i, L_{i+1} \) and \( L_{i+2} \). But we require an additional condition, every vertex in the total dominating set \( T \) must be adjacent to some vertex of \( T \).

III. MAIN RESULT

Theorem : The total domination number of Butterfly graph \( BF(n) \) is
\[
\gamma_t(BF(n)) \begin{cases} 
2^2 & \text{for } n = 2 \\
2^3 & \text{for } n = 3 \\
(2k+r) 2^{n-1} & \text{for } n = 4k+r, r = 0,1, \text{ and } k \in Z^+ \\
(k+1) 2^n & \text{for } n = 4k+r, r = 2,3, \text{ and } k \in Z^+. 
\end{cases}
\]

The butterfly graphs \( BF(2) \) and \( BF(3) \) have a special structure. \( BF(2) \) is the only butterfly graph having parallel edges and \( BF(3) \) is the only butterfly graph having triangles. So we give the total domination number of these two graphs separately. We find total domination number of \( BF(4) \) which is a 4-regular graph without triangles and then using Special Recursive construction of \( BF(4k) \) the results for total domination number of \( BF(4) \) are extended for \( BF(4k) \). Later using the Recursive construction 1 we find total domination numbers of \( BF(4k+r) \) for \( r = 1, 2, 3 \).

Case 1 : \( n = 2 \).

Consider the graph \( BF(2) \). We know that \( \gamma(BF(2)) = 2 \). Without loss of generality, we assume that a dominating set of \( BF(2) \) be \( D = \{(0; m_1), (1; m_2)\} \), where \( m_1, m_2 \in \{0, 1, 2, 3\} \) and \( m_1 + m_2 = 3 \). Possible values for \( m_1 \) and \( m_2 \) are 0, 1, 2, and 3. From the definition of edges in \( BF(2) \) these pairs of vertices in \( D \) are not adjacent. Clearly \( D \) is not a total dominating set as the two vertices in \( D \) are not adjacent and hence does not dominate each other. To dominate the vertices of \( D \), let us include a new vertex \((k; t)\) into \( D \). If \( k = 0 \) then \((k; t)\) is not adjacent to \((0; m_1)\) and if \( k = 1 \) then \((k; t)\) is not adjacent to \((1; m_2)\). So there is no single vertex \((k; t)\) which dominates both \((0; m_1)\) and \((1; m_2)\) of \( D \). Thus a dominating set of cardinality 3 is not a total dominating set. Hence if a total dominating set \( T \) exists in \( BF(2) \) then \(|T| > 3\).

Now \((0; m_1)\) is adjacent to 3 vertices of level \( L_1 \), so we include any one of these 3 vertices into \( D \) say \((1; m_2)\) where \( m_1 \neq m_2 \). Again vertex \((1; m_2)\) is adjacent to 3 vertices of \( L_0 \), so we include any one of these three vertices into \( D \) say \((0; m_3)\) where \( m_3 \neq m_2 \). Thus the vertices \((1; m_2)\) and \((0; m_3)\) are adjacent and similarly the vertices \((1; m_3)\) and \((0; m_4)\) are adjacent.

\[
T = \{(0; m_1), (1; m_2), (1; m_3), (0; m_4) / m_1 \neq m_3 \text{ and } m_2 \neq m_4\}
\]

Clearly \( D \subseteq T \). So \( T \) is a dominating set. Also vertices \((1; m_2)\) and \((0; m_3)\) dominate each other and similarly the vertices \((1; m_3)\) and \((0; m_4)\) dominate each other. It follows that \( T \) is a total dominating set of \( BF(2) \).

Case 2 : \( n = 3 \).

Consider the graph \( BF(3) \). We know that \( \gamma(BF(3)) = 6 \). As shown in [6] any dominating set \( D \) of \( BF(3) \) is an independent set. So \( D \) can not be a total dominating set. By recursive construction we observe that \( BF(3) \) contains two copies of \( BF(2) \) and a level with 8 vertices. Let us consider the first copy of \( BF(2) \) along with the first four vertices of level \( L_2 \) as left part of \( BF(3) \) and the remaining part as right part of \( BF(3) \). Hence, by symmetric structure of butterfly graph, we need to add at least one vertex each from left part and right part of \( BF(3) \) to \( D \), in order to get a total dominating set. Thus \( \gamma(BF(3)) \geq 8 \). Now we present a construction of a total dominating set of cardinality 8 for \( BF(3) \). Consider a set of vertices from level \( L_k \) given by \( T_1 = \{(k; m_1), (k; m_2), (k; m_3) / m_1, m_2 \neq m_3 \} \).

Vertices in \( T_1 \) dominate all the vertices of \( L_k \) and \( L_{k+1} \). (Lemma 4.3 of [6]). Now to dominate the remaining vertices of \( L_{k-1} \), we select vertices from \( L_{k-1} \) or \( L_{k+2} \). A vertex \((k-1; t)\) of \( L_{k-1} \) is adjacent to two vertices \((k; t)\) and \((k; s)\) where \(|t-s| = 2^k \). So only one of these vertices is in \( T_1 \) and the other is in \( L_0 \setminus T_1 \). (So \( k-1; t \) dominates one vertex of \( L_{k-1} \). Thus every vertex of \( L_{k-1} \) dominates one vertex of \( T_1 \) and one vertex of \( L_{k-1} \). So we need to select 4 vertices from \( L_{k-1} \) to dominate 4 vertices of \( L_{k-2} \). Let \( T_2 = \{(k-1; t_1), (k-1; t_2), (k-1; t_3) / t_i - t_j \neq 2^k \} \). Take \( T = T_1 \cup T_2 \). So \(|T| = 4 + 4 = 8 \).

Now vertices of \( T_1 \) dominate all the vertices of \( L_{k-1} \) and \( L_{k+1} \) and vertices of \( T_2 \) dominate all the vertices of \( L_{k-2} \). So vertices of \( T \) dominate all the vertices of \( BF(3) \). As vertices of \( T_1 \) are adjacent to vertices of \( T_2 \), it follows that \( T \) is a total dominating set of \( BF(3) \). Hence, \( \gamma(BF(3)) = 8 \). As \( \gamma(BF(3)) \geq 8 \), it follows that \( T \) is a minimal total dominating set of \( BF(3) \).

Case 3 : \( n = 4 \).

Consider the graph \( BF(4) \). We know that \( \gamma(BF(4)) = 16 \). As every total dominating set is dominating set, \( \gamma(BF(4)) \geq 16 \). Consider a dominating set of \( BF(4) \) given by \( D = D_1 \cup D_2 \), where the two sets \( D_1 \) and \( D_2 \) are \( D_1 = \{(1; m_1), (1; m_2) \ldots (1; m_8) / m_1, m_2 \neq m_3 \} \) and \( D_2 = \{(2; s_1), (2; s_2) \ldots (2; s_8) / s_1, s_2 \neq s_3 \} \).

Clearly from the definition of adjacency of vertices in two consecutive levels, the vertices in \( D_1 \) dominate all the vertices of level \( L_0 \) and \( L_2 \). As vertices of
D₂ belong to L₂, it is clear that a vertex in D₁ dominates a vertex in D₂ and vice-versa. So this set D dominates all the vertices of BF(4) and hence D is a total dominating set of BF(4). Since |D| = 16, D is a minimum total dominating set of BF(4). Therefore γ₁(BF(4)) = 16.

**Case 4: n = 4k**

We prove the result for BF(4k) by using the Principle of Mathematical Induction.

**Step 1:** Let k = 1. So n = 4. For k = 1, from theorem 3, the result is true for BF(4).

**Step 2:** Let us assume that the result is true for k = t. We prove the result for k = t+1. Consider the graph BF(4(t+1)). From Special Recursive Construction of BF(4k), BF(4t+4) is decomposed into 16 copies of BF(4t) without wings and the last 4 levels form a pattern with 2⁴ vertex groups, where each vertex group has 2⁴ vertices. From the induction hypothesis the result is true for BF(4t).

Consider a total dominating set S₁ of cardinality t x 2⁴ of BF(4t). The last 4 levels of BF(4t+4) form a pattern of BF(4) which is isomorphic to BF(4) without wings. Consider a total dominating set T₁ with 16 vertex groups for this pattern of BF(4) formed between the last 4 levels where each vertex group has 2⁴ vertices.

Suppose we get a total dominating set of cardinality less than 16 x 2⁴ for the pattern of BF(4). Since pattern of BF(4) is isomorphic to the graph BF(4) without wings, this assumption gives that we can get a total dominating set for BF(4) of cardinality less than 2⁵, which is a contradiction as γ₁(BF(4)) = 2⁵.

Let T = \( \bigcup_{i=1}^{16} S_i \cup T_1 \).

For i = 1, 2 ... 16, each Sᵢ is a minimum total dominating set of BF(4t) between levels L₀ to L₄t and T₁ is a minimum total dominating set of pattern of BF(4) between L₄t to L₄t+1. Thus all the Sᵢ’s and T₁ are disjoint sets. Hence T is a minimum total dominating set of BF(4t+4) whose cardinality is |T| = 16 x t x 2⁴ + 16 x 2⁴ = t x 2¹⁴ + 2⁴ = (t+1) 2¹⁴.

So the result is true for k = t+1. Hence by the Principle of Mathematical Induction the result is true for all positive integers k. Thus γ₁(BF(4k)) = k x 2⁴.

**Case 5: n = (4k + r) r = 1, 2, 3**

By Recursive Construction 1, there are 2⁴ copies of BF(4k) and r levels with 2⁴r vertices in BF(4k+r). To get a minimum total dominating set for BF(4k+r), first we include the vertices in a minimum total dominating set of disjoint copies of BF(4k) in first 4k levels. Now for total domination of vertices, we include vertices into T from last r levels as follows:

For r = 1, we choose 2⁻¹ vertices from level L₄ₗ to dominate all the vertices of the last level L₄ₗ.

For r = 2, we choose 2⁻¹ vertices from levels L₄ₗ and L₄ₗ to dominate all the vertices of the last two levels L₄ₗ and L₄ₗ.

For r = 3, we choose 2⁻¹ vertices from levels L₄ₗ and L₄ₗ to dominate all the vertices of the last three levels L₄ₗ, L₄ₗ+1 and L₄ₗ+2.

These vertices together with the minimum total dominating sets of r copies of BF(4k) form a minimum total dominating set of BF(4k+r). Combining these cases we get that the cardinality of minimum total dominating set for BF(4k+r), |D| =

\[
\begin{align*}
\gamma_1(BF(4k)) + 2^{4k} + 2^{4k} + 2^{4k} &= 2(k+1) 2^{4k} & \text{if } r=1 \\
2^{4k} + 2^{4k+r} + 2^{4k+2} + 2^{4k+3} &= (k+1) 2^{4k+3} & \text{if } r=2 \\
2^{4k} + 2^{4k+r} + 2^{4k+2} + 2^{4k+3} &= (k+1) 2^{4k+3} & \text{if } r=3
\end{align*}
\]

Thus we get \( \gamma_1(BF(4k+r)) = (2k+r) 2^{4k} \) for \( r = 0,1 \) and \( k \in Z^+ \).

One such choice is illustrated in the figure below for the graph BF(5) in which bold lines show domination of the vertices from dominating set D and thin lines show \( V \setminus D \) vertices domination.

![Figure 3: A Total dominating set for BF(5)](image)

**IV. CONCLUSION**

The total domination number of BF(2) is 2² and BF(3) is 2³. For n > 3 the total domination number is same as the domination number of BF(n). But every minimal dominating set is not minimal total dominating set. Only particular choices, form minimal total dominating sets. Thus \( \gamma_1(BF(n)) = \gamma(BF(n)) \) for n > 3.

**REFERENCES**