\(\alpha_0\)-Graph of a Graph

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Abstract:
The minimum covering sets of a graph may be used to create a new graph. The vertices of the new graph are of course the minimum covering sets of the original graph and two such vertices will be adjacent if in the corresponding minimum covering sets, one can move from one of them to other by deletion of a vertex and addition of another vertex. This new graph is called the \(\alpha_0\)-graph of the given graph. This paper makes a study of \(\alpha_0\)-graph of a graph.

Keywords: Covering sets, \(\alpha_0\)-graph of a graph.

I. INTRODUCTION

In [1] Dr.N.Sridharan introduced the concept of \(\gamma\)-graph. In [7], [8] Dr.V.Swaminathan, A.P.Pushpalatha and et al made a study of \(\beta_0\)-graph and \(\omega\)-graph of a given graph. The minimum covering sets of a graph may be used to create a new graph. The vertices of the new graph are of course the minimum covering sets of the original graph and two such vertices will be adjacent if in the corresponding minimum covering sets, one can move from one of them to other by deletion of a vertex and addition of another vertex. In this paper, we study the concept of \(\alpha_0\)-graph of a graph.

II. DEFINITIONS AND PROPERTIES

Let \(G=(V,E)\) be a simple, finite, and undirected graph. We follow the notation & terminology given by Harary[4].

Definition: 2.1
A set of vertices which covers all the edges in a graph \(G\) is called a vertex cover for \(G\). The vertex covering number \(\alpha_0(G)\) is the minimum cardinality of a vertex covering set. We call a set of vertices a \(\alpha_0\)-set if it is a vertex covering set with cardinality \(\alpha_0(G)\).

Definition: 2.2
Let \(G\) be a graph. Let \(H\) be a graph whose vertex set is the set of all covering sets of \(G\) and two vertices \(u, v \in H\) representing covering sets \(S_a, S_b\) respectively are adjacent iff \(S_a=(S_a-\{u\})\cup\{v\}\) where \(u\in S_a, v \notin S_a, i\neq j\). \(H\) is called the \(\alpha_0\)-graph of \(G\) and is denoted by \(\alpha_0(G)\).

Example: 2.3
\(\alpha_0(C_4) = K_2^2: \)
\(\{u_1, u_3\}, \{u_2, u_4\}\) are the \(\alpha_0\) sets of \(C_4\). Then the \(\alpha_0(C_4) = K_2^2\)

It is easy to find the \(\alpha_0\)-graph for several classes of graphs.

\(\alpha_0\)-graph for some standard graphs
1. \(\alpha_0(K_n) = K_n\)
2. Suppose \(G\) has a unique minimum covering set. Then \(\alpha_0(G) = K_1\). Therefore \(\alpha_0(K_w, w) = K_1\)
3. \(\alpha_0(K_m, n) =\) \(\begin{cases} \overline{K_2} & \text{if } m \geq 2 \text{ and } m = n \\ K_1 & \text{if } m \neq n \end{cases}\)
4. \(\alpha_0(P_n) =\) \(\begin{cases} P_{k+1} & \text{if } n = 2k, k \geq 2 \\ C_{n-1} & \text{if } n \text{ is odd} \end{cases}\)
5. \(\alpha_0(C_n) =\) \(\begin{cases} \overline{K_2} & \text{if } n \text{ is odd} \\ C_n & \text{if } n \text{ is even} \end{cases}\)
6. \(\alpha_0(W_n) =\) \(\begin{cases} \overline{K_2} & \text{if } n \text{ is odd} \\ C_{n-1} & \text{if } n \text{ is even} \end{cases}\)
7. \( \alpha_0(H) = K_1 \), where \( H \) is the subdivision of \( K_{1,n} \).
8. \( \alpha_0(K_n^*) = K_1 \).

**Observation: 2.4**

1. If \( G \neq K_n \) has distinct “p” \( \alpha_0 \)-sets, then \( \alpha_0(G) = pK_1 \).
2. If \( G \) has a unique \( \alpha_0 \)-set, then \( \alpha_0(G) = K_1 \).
3. \( |V(\alpha_0(G))| \leq |V(G)| \).

**Remark: 2.5**

More than one graph may have the same \( \alpha_0 \)-graph. For example \( C_{2n}, W_n \) and \( K_{m,m} \) produce the same \( \alpha_0 \)-graph, namely \( K_2 \).

**Theorem: 2.6**

If any pair of distinct \( \alpha_0 \)-sets differ in exactly one place, then \( \alpha_0(G) = K_p \), where \( p \) is the number of distinct \( \alpha_0 \)-sets of \( G \).

**Proof**

Let \( G \) be a graph with \( \alpha_0 \)-sets namely \( S_1, S_2, \ldots, S_p \). Suppose \( S_i, S_j \) \( (1 \leq i, j \leq p/i \neq j) \) are distinct in exactly one place, then \( S_i \) and \( S_j \) are adjacent in \( \alpha_0(G) \) for all \( i, j \). Then the resultant graph is a complete graph \( K_p \).

**Theorem: 2.7**

\[ \alpha_0(K_n^* K_k) = K_{1,n} \].

**Illustration**

![Graph Illustration](image_url)

The \( \alpha_0 \)-sets of \( K_3^* K_1 \) are \( \{ u_1, u_2, u_3 \} \) (say \( S_1 \)), \( \{ u_1, u_3, u_1' \} \) (say \( S_2 \)), \( \{ u_1, u_2, u_1' \} \) (say \( S_3 \)) and \( \{ u_1, u_2, u_1 \} \) (say \( S_4 \)). Hence the \( \alpha_0 \)-graph of \( K_3^* K_1 \) is given by

**Remark: 2.8**

Any star \( K_{1,n}, n \geq 1 \) is a graph of some graph.

**Definition: 2.9**

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be any two graphs. Then their Cartesian Product \( G_1 \square G_2 \) is defined to be the graph whose vertex set is \( V_1 \cup V_2 \) and edge set is \( \{(u_1, v_1), (u_2, v_2)\} \) either \( u_1 = u_2 \) and \( v_1, v_2 \in E_2 \) or \( v_1 = v_2 \) and \( u_1, u_2 \in E_1 \).

**Theorem: 2.10**

For the grid graph \( P_m \square P_n \), where \( \square \) denotes the Cartesian product of \( P_m \) and \( P_n \),

\[ \alpha_0(P_m \square P_n) = \begin{cases} K_2, & \text{if } mn \equiv 0 \pmod{2} \\ K_1, & \text{otherwise} \end{cases} \]

**Proof**

If \( mn \equiv 0 \pmod{2} \), then \( P_m \square P_n \) has exactly two distinct minimum \( \alpha_0 \)-sets of cardinality \( \frac{mn}{2} \). Hence \( \alpha_0(P_m \square P_n) = K_2 \) if \( mn \equiv 1 \pmod{2} \), then \( P_m \square P_n \) has a unique \( \alpha_0 \)-set of cardinality \( \frac{mn}{2} \). Hence \( \alpha_0(P_m \square P_n) = K_1 \).

**Illustration: 2.11**

![Grid Graph Illustration](image_url)

The minimum covering set is \( \{2, 4, 6, 8, 10, 12, 14\} \). Since \( P_3 \square P_2 \) has only one minimum covering set, \( \alpha_0(P_3 \square P_2) = K_3 \).

**Theorem: 2.12**

If \( G \) is a totally disconnected graph, then \( \alpha_0(G) = K_1 \).

**Proof**

Let \( G \) be a totally disconnected graph with the vertex set \( \{ u_1, u_2, u_3, \ldots, u_k \} \). Then \( \{ u_1, u_2, u_3, \ldots, u_k \} \) is a unique \( \alpha_0 \)-set of \( G \). By observation, \( \alpha_0(G) = K_1 \).

**Theorem: 2.13**

Let \( G \) be a disconnected graph with two complete components \( K_a \) and \( K_m \). Then \( \alpha_0(G) = K_a \square K_m \).

**Proof**

Let \( V(K_a) = \{ u_1, u_2, u_3, \ldots, u_k \} \) and \( V(K_m) = \{ v_1, v_2, v_3, \ldots, v_m \} \). Then \( \alpha_0 \)-sets of \( G \) are \( \{(u_i, v_j)/1 \leq i \leq \)}
n, 1 ≤ j ≤ m}. Fix i, (u_i, v_j) is adjacent to (u_i, v_j') j = j'. These m points form a complete graph K_m. Fix j, (u_i, v_j) is adjacent to (u_i', v_j'), i = i'. These n points form another graph K_n. We get the resultant graph K_n ⊗ K_m.

**Theorem 2.14**

\[ \alpha_0(K_n - \{e\}) = K_1, n ≥ 3. \]

**Proof**

Let G be a graph obtained from K_n by deleting any one edge. Hence in K_n - {e} = G, only two vertices are of degree n-2 and others are having degree n-1. Hence \( \alpha_0(G) = n - 2 \) and G has a unique \( \alpha_0 \)-set. Therefore \( \alpha_0(K_n - \{e\}) = K_1. \)

**Theorem 2.15**

\[ \alpha_0(K_n - \{u\}) = K_{n-1}, n ≥ 2. \]

**Proof**

The deletion of any vertex from K_n is nothing but K_{n-1}. Since \( \alpha_0(K_n) = K_n \forall n, \alpha_0(K_n - \{u\}) = K_{n-1}, n ≥ 2. \)

**Theorem 2.16**

If \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs with \[ |V(G_2)| ≤ |V(G_1)| \] and \( G_1 \) has a unique \( \alpha_0 \)-set, then \( \alpha_0(G_1 + G_2) = K_1. \)

**Proof**

Let \( S_1 \) and \( S_2 \) be the \( \alpha_0 \)-sets of \( G_1 \) and \( G_2 \) respectively and \( S_3 \) is unique \( \alpha_0 \)-set of \( G_4 \). Then the \( \alpha_0 \)-set of \( G_1 + G_2 \) is \( S_1 \cup V(G_2) \). It is obvious that \( S_1 \cup V(G_2) \) is a unique \( \alpha_0 \)-set of \( G_1 + G_2 \). Therefore, \( \alpha_0(G_1 + G_2) = K_1. \)

**REFERENCES**


