An Upper Bound to the Number of Conjugacy Classes of Non-Abelian Nilpotent Gathering or Groups

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ABSTRACT

The quantity of conjugacy classes of symmetric group, dihedral group and some nilpotent groups is gotten. Up to this point, it has not been gotten for every single nilpotent groups. Despite the fact that there are some lower bound to this esteem, there is no non-minor upper bound. This paper intends to explore an upper bound to this number for all limited nilpotent groups. Also, the correct number of conjugacy classes is found for a specific case of non-abelian nilpotent groups.

Keywords— Conjugacy Class, Center, Nilpotent Class

I. INTRODUCTION

Characterizing limited nilpotent groups is a troublesome undertaking, however there are some nilpotent groups that have been ordered. The characterization of conjugacy classes for specific groups, for example, symmetric groups also, dihedral groups have been resolved, see (Erdos and Turan, 1968; Pantea, 2004; Banti, 2005; Héthelyi et al., 2011; Al-Hasanat and Abdullah, 2012). In this exploration, we will mean \( c_l_G \) as the quantity of conjugacy classes of a groups G.

The travail aggregate has just a single conjugacy class where an abelian amass G has \(|G|\) conjugacy classes. Subsequently, \( 1 \leq c_l_G \leq |G| \) for any limited group G. Erdos and Turan (1968) demonstrated that G, \( c_l_G > \log \log 2 |G| \) for any limited group G. An enhanced lower bound was gotten by Sherman (1979), he demonstrated that in the event that G is nilpotent of class c, at that point \( c_l_G > c(|G|^{1/c} - 1) + 1 \), which shows the impact of the nilpotency class c on \( c_l_G \).

The quantity of conjugacy classes was found for limited nilpotent groups in Lopez (1985), it is gotten as a capacity of the requests of specific subgroups. The conjugacy classes of p- groups, with an abelian subgroup of record p, with \( |Z(G)| = |G| \), were contemplated in (Pantea, 2004). Zapirain (2011) set up a lower bound for the quantity of conjugacy classes of limited nilpotent groups utilizing the group size duplicate by a certain steady c. As of late, Ahmad et al. (2012) got general equations to locate the quantity of conjugacy classes of 2-generator p- groups of class 2.

The absence of using just a lower bound for \( c_l_G \) would make the recognizable proof of this esteem a troublesome undertaking, particularly for groups of enormous size. Discretionarily, let \( L = c(|G|^{1/c} - 1) + 1 \) be the lower bound acquired by Sherman (c is the nilpotency class of G), at that point \( L \leq c_l_G \leq |G| \) (the paltry upper bound \(|G|\) will wander for vast size groups).

In this investigation, another approach has been trailed by a particular course. In other words, the majority of the past inquire about got bring down limits for the estimation of \( c_l_G \) or acquired this incentive for specific groups and not for all non-abelian nilpotent groups. This work means to enhance the conventional upper bound \(|G|\) to another shut one. The estimation conveyed the count of a sum u as a capacity of the requests of the group G and its inside \( Z(G) \), together with the nilpotency class c. The substitution of the estimation of u rather than \( |G| \) as an upper bound will confine the esteem \( c_l_G \) with more united limits.

The paper is organized as takes after. In the following segment we settle our documentation and build up some preparatory comes about. In segment 3 the meaning of the conjugacy classes is given. In segment 4 layout our methodology for evaluating the quantity of conjugacy classes. What's more, we demonstrate the primary outcomes and show a few cases. In the last area, we portray the examination comes about where the exploration conclusions were given.
Our documentation is genuinely standard. All through, all bunches are limited. For a group $G$, $e$ is the character component furthermore, $Z(G)$ means the focal point of $G$. Review the commutator of $x, y \in G$ is given by $[x, y] = x - 1y - 1xy$. For any subgroups $A$ and $B$ of a gathering $G$, the commutator subgroup $[A, B]$ is the group $\{[a, b] | a \in A, b \in B\}$. The determined subgroup of $G$ is:

$$G' = [G, G] = \langle [xy] | x, y \in G \rangle$$

The immediate result of the groups $(G, \ast)$ and $(H, \ast)$ is $\langle (g, h) | g \in G, h \in H \rangle$ and meant by $G \rtimes H$ with:

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \ast g_2, h_1 \ast h_2)$$

The lower focal arrangement of a group $G$ is:

$$G = \gamma_0(G) \geq \gamma_1(G) \geq \cdots \geq \gamma_r(G) \geq \cdots$$

where, $\gamma_i(G) = [\gamma_{i-1}(G), G]$.

**Definition 1**

A groups $G$ is called nilpotent if there exists $c$ in the bring down focal arrangement with the end goal that $\gamma_c(G) = \{e\}$ and the littlest such esteem $c$ is the class of nilpotency.

**III. THE CONJUGACY CLASSES**

The components $x$ and $y$ in $G$ are said to be conjugate, on the off chance that there exists a component $g \in G$ with the end goal that $x = gyg^{-1}$. The arrangement of all $y$ in $G$, which are conjugate to $x$ is the conjugacy class of $x$ and signified by $CC_G(x)$. The conjugacy classes of $G$ are $\{CC_G(x) | x \in G\}$ and the request of this set is $cl_G$.

The conjugacy connection is a proportionality connection. In the event that no unmistakable components of $G$ create a similar conjugacy classes, at that point $cl_G = |G|$. Note that $CC_G(e) = \{e\}$, hence $1 \leq cl_G \leq |G|$. The equity holds that if $G$ is the insignificant group $G = \{e\}$ at that point $cl_G = 1$, or if $G$ is abelian group at that point $cl_G = |G|$. Tragically, for groups of vast request, these limits wander and we watch that the depiction of $cl_G$ turns out to be extremely troublesome.

**IV. THE NUMBER OF CONJUGACY CLASSES**

Unmistakably $CC_G(a) = \{a\}$ for every one of the $a \in Z(G)$. In this way, $cl_G = |Z(G)| + |CC_G(x) | x \in GZ(G)\}$, where the estimation of the second term is exceptionally troublesome. The next hypothesis will demonstrate an upper bound for these estimations utilizing just three known esteems, which are; the request of the group, the request of the group focus and the nilpotency class.

**Theorem 2**

Give $G$ a chance to be a limited group of nilpotency class $c$ and $Z(G)$ be the focal point of $G$. Let $u = |G| - r$, where $r = \left\lceil \frac{|G| - |Z(G)|}{c} \right\rceil$. At that point:

$$cl_G \leq u$$

**Proof**

Give $G$ a chance to be a group of nilpotency class $c$ and let $n(G)$ indicates the quantity of conjugacy classes $CC_G(x)$ for all $x \in G \setminus Z(G)$. At that point:

$$cl_G = |Z(G)| + n(G)$$

As $G$ acts by conjugation on the arrangement of all components $x \in G \setminus Z(G)$, at that point each circle has length $\geq 2$ and the number of such circles is $n(G)$. Subsequently $n(G) \leq \frac{|G|}{2} |G| - |Z(G)| \rangle$. On the off chance that $G$ is an abelian group, at that point $c = 1, |G| = |Z(G)|$ and $n(G) = 0$. Hence, "Disparity 1" holds with equity. In the event that $G$ is a group of nilpotency class $c > 1$, at that point $1 > \frac{c-1}{c} \geq \frac{1}{2}$. Hence:

$$n(G) \leq \frac{|G| - |Z(G)|}{c} \leq \frac{c-1}{c} (|G| - |Z(G)|)$$

Utilizing Equation 2. It takes after that:

$$cl_G = |Z(G)| + \frac{c-1}{c} (|G| - |Z(G)|)$$

$$= |Z(G)| + 1 - \frac{1}{c} (|G| - |Z(G)|)$$

$$\leq |G| - \frac{|G| - |Z(G)|}{c}$$

The estimation of $u$ as it delineated in the past hypothesis is $u = |G| - r$,

$$\text{with } r = \left\lceil \frac{|G| - |Z(G)|}{c} \right\rceil$$

Realizing that, the set $Z(G)$ is a subgroup of $G$, at that point $|G| \geq |Z(G)|$. What's more, the nilpotency class $c$ is the list of the lower focal arrangement of $G$, so $c$ is entirely positive number. In this way, this suggests, $0 \leq r \leq |G|$. Thus, $0 < u = |G| - r \leq |G|$.  

**Example 3**

The accompanying table clarifies the utilization of Theorem 2. It ought to be tried by Table 1. All groups in Table 1 are nilpotent groups. Applying Hypothesis 2 on this table to get the last section, it shows up that $cl_G \leq u$ for all groups. On the off chance that $u$ is decreased more, that is $u < |G| - r$, at that point $u$ won't likely function as an
upper bound to $cl_G$ for all groups recorded in this table. As the cyclic assemble $c_n$ is an abelian group, at that point $cl_G = |G| = n$. Along these lines, if $u < |G| = n$, at that point $cl_G = n > u$, which suggests that $u$ is not an upper bound to $cl_G$.

**Corollary 4**

Give G a chance to be a limited group of nilpotency class 2. On the off chance that $|G'| = 2$, at that point:

$$cl_G = \frac{1}{2}(|G| - |Z(G)|)$$

**Proof**

Give G a chance to be a limited group of nilpotency class 2, with $|G'| = 2$. At that point:

$$y^2x = y[y, x^{-1}][x^{-1}, y^{-1}]yx = y(y^{-1}xyx^{-1})(xyx^{-1}y^{-1})yx = xy^2$$

for all $x, y \in G$ and the request of $[x, y]$ in $G$ separates the requests of $xZ(G)$ and $yZ(G)$ in $G/Z(G)$. So if $n(G)$ is the number of conjugacy classes of all $x \in G / Z(G)$, at that point each circle has length precisely 2 and the quantity of such circles is $n(G)$.

Subsequently $n(G) = \frac{|G| - |Z(G)|}{2}$.

At that point, the claim is taken after by utilizing "Condition 2".

**Example 5**

The following tables close the required estimations for a few groups $G$ to demonstrate the connection amongst $u$ and $cl_G$ and to exhibit the utilization of Corollary 4. It ought to be taken after by Table 2 and 3. Table 2 contains just groups of nilpotency class $c = 2$ what's more, $|G'| = 2$; utilizing Corollary 4 suggests that $cl_G = u$.

Table 1. Utilizing Theorem 2 to discover $cl_G$ for a few groups of nilpotency class $c$

| Group Structure | $|G|$ | $|Z(G)|$ | $c$ | $cl_G$ | $u$ |
|-----------------|------|---------|----|--------|-----|
| $C_5 \times D_{22}$ | 160 | 10      | 4  | 55     | 123 |
| $C_5$           | n    | n       | 1  | n      | n   |
| $C_5 \times D_8$ | 40   | 10      | 2  | 25     | 25  |
| $D_{128}$       | 128  | 2       | 6  | 35     | 107 |

Table 2. The evaluated upper bound $u$ meets $cl_G$ for groups of nilpotency class 2 and $|G'| = 2$

| Group Structure | $|G|$ | $|Z(G)|$ | $|G'|$ | $c$ | $cl_G$ | $u$ |
|-----------------|------|---------|-------|----|--------|-----|
| $C_4 \times D_8$ | 32   | 8       | 2     | 2  | 2      | 20  |
| $C_8 \times C_8$ | 64   | 16      | 2     | 2  | 2      | 40  |
| $C_8 \times D_8$ | 128  | 32      | 2     | 2  | 2      | 80  |
| $C_{11} \times D_8$ | 88  | 22      | 2     | 2  | 2      | 55  |
| $C_{22} \times C_8$ | 64  | 16      | 2     | 2  | 2      | 40  |
| $C_2 \times D_8$ | 16   | 4       | 2     | 2  | 2      | 10  |
| $C_{12} \times D_8$ | 96  | 24      | 2     | 2  | 2      | 60  |

Table 3 contains groups of nilpotency class $c = 2$ yet $|G'| \neq 2$. So we can utilize just Theorem 2 to bound the number of conjugacy classes. In this way, $cl_G < u$.

**V. CONCLUSION**

Utilizing the assessed esteem $u$ will enhance the insignificant upper bound of $cl_G$, which is $|G|$ and make it shut to the correct an incentive for substantial groups. The estimation of $u$ depends on $|Z(G)|$ and the nilpotency class $c$ notwithstanding the principle factor $|G|$, this will demonstrate the impact of $c$ on $cl_G$, which makes $u$ the primary bound to utilize these qualities. The estimations meet a similar estimation of $cl_G$ for all nilpotent groups, notwithstanding abelian groups (utilizing Groups, Calculations and programming GAP). In expansion, in Corollary 4 we can locate the correct measure of $cl_G$, utilizing as basic as less terms and estimations than the typical estimation.

**REFERENCES**