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Bayesian Estimation of Continuous Change Point in Exponential Distribution

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ABSTRACT

A patient is surviving according to exponential life time distribution. If at some unknown point of time τ the Stress function changes then the life time distributionalsochanges. Exponential change point life time model is proposed. Bayes estimator of unknown change point τ is obtained under asymmetric loss function.

***Keywords*—**Variables, Function, GAMA

I. INTRODUCTION

A component fails when the stress induced by the operating conditions exceeds the stress resisting capacity (strength) of the component.We consider strength is random variables since it depend on several manufacturing variables such as temperature, size, surface finish etc. Now, if the strength changes with time then the reliability and hence the whole model is changed. Considering these possibilities, we proposed and study Bayesian estimation of exponential change point model. Mayuri Pandya(2004) had studied the Bayesian analysis of the inverse weibull change point model considering continuous change point in strength.

II. CHANGE POINT MODEL CONSIDERING CHANGE IN STRENGTH

Let, stress s(t) be a time increasing function which changes at some unknown time τ , viz

$S(t) = \lambda t$; $t < \tau$

$= \lambda(\tau + \rho(t - \tau))$; $t \geq \tau$ (1)Let, we assume strength Y as an exponential random variable with mean value $1/\theta$, i.e the probability density function of Y is

$$g(y) = \theta e^{-y\theta} \quad ; \theta > 0, x > 0 \tag{2}$$

Then the stress strength model is given by,

$R_X(t)$ = Probability that component survive beyond time t

$$= \int_{S(t)}^{\infty} g(y)dy$$

$$= e^{-S(t)*\theta} \tag{3}$$

So we propose following change point model (Stress change point model) related with

The stress –strength model is:

$$R_x(t) = \begin{cases} e^{-(\lambda\theta)t} & t < \tau \\ e^{-(\lambda(\tau+\rho(t-\tau))\theta)t} & t \geq \tau \end{cases} \tag{4}$$

Hence,

$$1 - F_x(t) = \begin{cases} e^{-\alpha t} & t < \tau \text{ where } \alpha = \lambda\theta \\ Ke^{-\alpha t'} & t \geq T \text{ where } \alpha = \lambda(\tau + \rho(t-\tau))\theta \end{cases}$$

Thus upper equation will be,

$$f(X|\alpha) = \begin{cases} \alpha \exp[-\alpha x] & ,x<\tau \\ k\,\rho\,\alpha \exp[-\alpha x'] & ,x\geq\tau \end{cases}, \quad x>0$$

Where k is such that

$$1 = \int_0^{\tau} f(x)dx + \int_{\tau}^{\infty} f(x')dx'$$

Which gives

k=1

$1=\int_0^{\tau} \alpha \exp[-\alpha x]\, dx + k \int_{\tau}^{\infty} \rho\alpha \exp[-\alpha x']\, dx'$

Hence,

$$F(X \mid \alpha, \beta) = \begin{cases} \alpha \exp[-\alpha x] & , x<\tau \\ \rho\,\alpha \exp[-\alpha x'] & ,x\geq\tau , x>0 \end{cases}$$

Where,
$x' = \tau + \rho(x - \tau)$ $\alpha = \lambda\theta$

The likelihood function of α and τ given $\underline{X} = (X_1, X_2, \ldots\ldots, X_n)$ is

$L(\alpha, \tau \mid \underline{X}) = \prod_{k=1}^{n} \{\alpha e^{-\alpha x_i}\}^{\epsilon_i} \{\rho\alpha e^{-\alpha x'_i}\}^{1-\epsilon_i}$

$= \rho^{n-d_1(\tau)} \alpha^n \exp[-\alpha \sum_{i=1}^{n} x_i \epsilon_i] \exp[-\alpha \sum_{i=1}^{n} x'_i (1-\epsilon_i)]$

Where,
$d_1(\tau) = \sum_{i=1}^{n} \epsilon_i = d_1(\tau \mid \underline{x})$
$A_1(\tau) = \sum_{i=1}^{n} x_i \epsilon_i = A_1(\tau \mid \underline{x})$
$A_2(\tau) = \sum_{i=1}^{n} x'_i (1-\epsilon_i) = A_2(\tau \mid \underline{x})$

$$\varepsilon_i = \begin{cases} 1, & if \quad x_i < \tau \\ 0, & if \quad x_i \geq \tau \end{cases} \quad (5)$$

III. POSTERIOR DENSITIES USING INFORMATIVE PRIOR IN MODEL-1 (INVERTED GAMMA PRIOR)

In this section, we derive marginal posterior density of τ using informative prior.We suppose the marginal prior distributions of α be inverted gamma distribution viz,

$g(\alpha) = \frac{b^a}{\Gamma a} \alpha^{-a-1} \exp(-b/\alpha)$

; $a = (\mu/\sigma)^2 + 2$
$b = \mu[(\mu/\sigma)^2 + 1]$ (6)

If the prior information is given in terms of the prior mean and standard deviation then hyper parameters can be obtained by solving the equations,

$$1 + \frac{2}{b} = [1 + 1/\ b]^d$$

$$a = \frac{\ln[\mu]}{\ln[b/(b+1)]}$$

For unknown change point τ, we assume that it takes one of the n observed values $x_1, x_2, \ldots\ldots, x_n$ and taking discrete values with prior probability i=1,2,…..,n. Then the joint prior density of α and τ_i , $g_1(\alpha, \tau_i)$ is

$g_1(\alpha, \tau_i) = g_1(\alpha \mid \tau_i = x_i) \prod_0(\tau_i = x_i)$

$= \frac{b^a}{\Gamma a} \alpha^{-a-1} \exp(-b/\alpha) \prod_0(\tau_i = x_i)$

$= K_1 \alpha^{-a-1} \exp(-b/\alpha)$ (7)

Where,

$K_1 = \frac{b^a}{\Gamma a} \prod_0(\tau_i = x_i)$

Joint posterior density of α and τ_i, $g_1(\alpha, \tau_i \mid x)$, is given by,

$g_1(\alpha, \tau_i \mid x) = L(\alpha, \tau_i \mid \underline{x})\, g(\alpha, \tau_i) / h_1(\underline{x})$

$= \rho^{n-d_1(\tau)} \alpha^n \exp(-\alpha A_1) \exp(-\alpha A_2) \frac{b^a}{\Gamma a} \alpha^{-a-1} \exp(-b/\alpha) \prod_0(\tau_i = x_i) / h_1(\underline{x})$

$= \rho^{n-d_1(\tau)} \frac{b^a}{\Gamma a} \prod_0(\tau_i = x_i)\, \alpha^{n-a-1} \exp(-\alpha(A_1 + A_2)) \exp(-b/\alpha) / h_1(\underline{x})$ (8)

Where,

$h_1(\underline{x}) = \sum_{i=1}^{n} \int_0^\infty \rho^{n-d_1(\tau)} \frac{b^a}{\Gamma a} \prod_0(\tau_i = x_i)\, \alpha^{n-a-1} \exp(-\alpha(A_1 + A_2)) \exp(-b/\alpha)\, d\alpha$

$= \sum_{i=1}^{n} \int_0^\infty K_2\, \alpha^{n-a-1} \exp(-\alpha(A_1 + A_2)) \exp(-b/\alpha)\, d\alpha$ (9)

$K_2 = \rho^{n-d_1(\tau)} \frac{b^a}{\Gamma a} \prod_0(\tau_i = x_i)$

If [Re(A_3)>0 && Re(b)>0, $2\, A_3^{\frac{a-n}{2}}\, ({}^1/_b)^{\frac{a-n}{2}}$

Bessel k[a-n, $2\sqrt{A_3}\sqrt{b}$; $\int_0^\infty \alpha^{a+n-1} \exp(-\alpha A_3 - \frac{b}{\alpha})\, d\alpha$ (10)

Hence, marginal posterior density of chang point τ_i , $g_1(\tau_i = x_i \mid \underline{x})$ is given by,

$g_1(\tau_i = x_i \mid \underline{x}) = \int_0^\infty g_1(\alpha, \tau_i \mid x)\, d\alpha$

$= \int_0^\infty \rho^{n-d_1(\tau)} \frac{b^a}{\Gamma a} \prod_0(\tau_i = x_i)\, \alpha^{n-a-1} \exp(-\alpha(A_1(\tau) + A_2(\tau))) \exp(-b/\alpha) / h_1(\underline{x})\, d\alpha$

$= \rho^{n-d_1(\tau)} \frac{b^a}{\Gamma a} \prod_0(\tau_i = x_i) \int_0^\infty \alpha^{n-a-1} \exp(-\alpha(A_1(\tau) + A_2(\tau))) \exp(-b/\alpha) / h_1(\underline{x})\, d\alpha$

$= \rho^{n-d_1(\tau)} \frac{b^a}{\Gamma a} \prod_0(\tau_i = x_i) \{2\, (A_1(\tau) + A_2(\tau))^{\frac{a-n}{2}} (\frac{1}{b})^{\frac{a-n}{2}}$ (6)

Bessel K[a-n, $2\sqrt{A_1(\tau) + A_2(\tau)}\sqrt{b}$] / $h_1(\underline{x})$

; If [Re $(A_1(\tau) + A_2(\tau))$ >0 && Re(b)>0] (11)

USING NON-INFORMATIVE PRIOR

In this section, we derive marginal posterior density of τ using non-informative prior.

Sometimes no prior information or technical knowledge about the parameters are available then we take Non-informative prior. Let us consider such non-informative prior densities on α_1 and α_2 to be,

$g_2(\alpha) = \frac{1}{\alpha}$;α>0

$g_2(\alpha, \tau \mid \underline{x}) = g_2(\alpha \mid \tau_i = x_i) \prod_0(\tau_i = x_i)$

$g_2(\alpha, \tau) = \frac{\prod_0(\tau_i = x_i)}{\alpha}$

Now,

$L(\alpha, \tau \mid \underline{x})\, g_2(\alpha, \tau)$

$= \frac{\prod_0(\tau_i = x_i)}{\alpha} \rho^{n-d_1(\tau)} \alpha^n \exp(-\alpha \sum_{i=1}^{n} x_i \varepsilon_i) \exp(-\alpha \sum_{i=1}^{n} x_i' (1-\varepsilon_i))$ (12)

Joint posterior density of α and τ_i is given by

$g_2(\alpha, \tau_i \mid \underline{x}) = L(\alpha, \tau \mid \underline{x})\, g_2(\alpha, \tau_i) / h_2(\underline{x})$

Where,

$h_2(\underline{x}) = \sum_{i=1}^{n} \int_0^\infty L(\alpha, \tau \mid x)\, g_2(\alpha, \tau)\, d\alpha$ (13)

$$=\sum_{i=1}^{n}\int_0^\infty \frac{\prod_0(\tau_i=x_i)}{\alpha}\rho^{n-d_1(\tau)}\alpha^n \exp(-\alpha\sum_{i=1}^{n}x_i\,\varepsilon_i)\exp(-\alpha\sum_{i=1}^{n}x_i'(1-\varepsilon_i))\,d\alpha$$

$$=\sum_{i=1}^{n}\prod_0(\tau_i=x_i)\int_0^\infty \rho^{n-d_1(\tau)}\alpha^{n-1}\exp(-\alpha\sum_{i=1}^{n}x_i\,\varepsilon_i)\exp(-\alpha\sum_{i=1}^{n}x_i'(1-\varepsilon_i))d\alpha$$

$$=\sum_{i=1}^{n}\prod_0(\tau_i=x_i)\int_0^\infty \rho^{n-d_1(\tau)}\alpha^{n-1}\exp(-\alpha A_1(\tau))\exp(-\alpha A_2(\tau))\,d\alpha$$

$$=\sum_{i=1}^{n}K_3\int_0^\infty \alpha^{n-1}\exp(-\alpha(A_1(\tau)+A_2(\tau)))\,d\alpha$$

Where, $K_3=\rho^{n-d_1(\tau)}\prod_0(\tau_i=x_i)$

$$=\sum_{i=1}^{n}K_3\frac{\Gamma n}{(A_1(\tau)+A_2(\tau))^n} \qquad (14)$$

Hence, marginal posterior density of change point τ_i, $g_2(\tau_i = x_i \mid \underline{x})$ is given by,

$g_2(\tau_i = x_i \mid \underline{x}) = \int_0^\infty g_2(\alpha,\tau \mid x)\,d\alpha$

$= \frac{1}{h_2(x)}\prod_0(\tau_i=x_i)\,\rho^{n-d_1(\tau)}\,\{\int_0^\infty \alpha^{n-1} \exp(-\alpha\,(A_1(\tau)+A_2(\tau)))\}\,d\alpha$

$=\frac{1}{h_2(x)}\prod_0(\tau_i=x_i)\,\rho^{n-d_1(\tau)}\frac{\Gamma n}{(A_1(\tau)+A_2(\tau))^n}$

$=\frac{\prod_0(\tau_i=x_i)\,\rho^{n-d_1(\tau)}I_1(\tau_i)}{\sum_{i=1}^{n}\prod_0(\tau_i=x_i)\,\rho^{n-d_1(\tau)}I_1(\tau_i)}$

Where, $I_1(\tau_i) = \frac{\Gamma n}{(A_1(\tau)+A_2(\tau))^n}$ (15)

marginal posterior density of α, given by g(α | x)

$g(\alpha \mid \underline{x}) = \int_0^\infty g(\alpha \mid x)\,d\alpha$

$= \alpha^{n-a-1}\frac{b^a}{\Gamma a}\exp(-b/\alpha) \sum_{i=1}^{n}\rho^{n-d_1(\tau)}$ $\exp(-\alpha\,(A_1(\tau)+A_2(\tau)))$(16)

The joint posterior density of α and $g_2(\alpha \mid x)$ is given by,

$g_2(\alpha \mid x) = \sum_{i=1}^{n} g_4(\alpha,\tau_i \mid x)$

$= \frac{1}{h_2(x)}\sum_{i=1}^{n}\prod_0(\tau_i=x_i)\,\rho^{n-d_1(\tau)}\,\alpha^{n-1}\exp(-\alpha\,(A_1(\tau)+A_2(\tau)))$(17)

IV. BAYES ESTIMATES

4.1 Using Informative Prior

In this section we obtain Bayes estimates of change point τ using informative prior for this model.Expected loss function, $E_1[L_1(\tau, d)]$ with respect to the posterior density , we get the Bayes estimate ${\tau_L}^*$ of τ using Linex loss function as,

${\tau_L}^* = \frac{-1}{q_1}\ln E_1[e^{-q_1\tau}]$

${\tau_L}^* = \frac{-1}{q_1}\ln\,[\sum_{i=1}^{n}e^{-q_1\tau_i}\rho^{n-d_1(\tau)}\frac{b^a}{\Gamma a}\prod_0(\tau_i=x_i)\,\{2\,(A_1(\tau)+A_2(\tau))^{\frac{a-n}{2}}(\frac{1}{b})^{\frac{a-n}{2}}$

Bessel K[a-n, $2\sqrt{A_1(\tau)+A_2(\tau)}\sqrt{b}$] / $h_1(\underline{x})$

; If [Re $(A_1(\tau)+A_2(\tau))$ >0 && Re(b)>0] (18)

Similarly, Bayes estimate ${\tau_E}^*$ of τ using General Entropy loss function as,

${\tau_E}^* = [\,E_2(\tau)^{-q_3}]^{-1/q_3}$

${\tau_E}^*$=[

$\int_0^\infty \alpha^{n-a-1}\,\alpha^{-q_3}\frac{b^a}{\Gamma a}\exp(-b/\alpha)\sum_{i=1}^{n}\rho^{n-d_1(\tau)}\exp(-\alpha(A_1(\tau)+$
$A2\tau]-1q3$ d α

=[
$\frac{b^a}{\Gamma a}\sum_{i=1}^{n}\rho^{n-d_1(\tau)}\int_0^\infty \alpha^{n-a-1}\,\alpha^{-q_3}\exp(-b/\alpha)\exp(-\alpha(A_1(\tau)+$
$A2\tau]d\alpha-1q3$

$= \frac{b^a}{\Gamma a}\sum_{i=1}^{n}\rho^{n-d_1(\tau)}\;2\;(A_1(\tau)+A_2(\tau))^{\frac{a+q_3-n}{2}}(\frac{1}{b})^{\frac{a+q_3-n}{2}}$

Bessel K[a+q_3-n, $2\sqrt{A_1(\tau)+A_2(\tau)}\sqrt{b}$]

; If [Re $(A_1(\tau)+A_2(\tau))$ >0 && Re(b)>0](19)

4.2 Using Non Informative Prior

${\tau_L}^{**} = \frac{-1}{q_1}\ln E_2[e^{-q_1\tau}]$

${\tau_L}^{**} = \frac{-1}{q_1}\ln\,[\sum_{m=1}^{n-1}e^{-q_1\tau_i}\prod_0(\tau_i=x_i)$ (15)
$\rho^{n-d_1(\tau_i)}I_1(\tau_i)/\sum_{m=1}^{n-1}\prod_0(\tau_i=x_i)$
$\rho^{n-d_1(\tau_i)}I_1(\tau_i)]$ (20)

${\tau_E}^{**} = [\,E_2(\tau)^{-q_3}]^{-1/q_3} = [\,\sum_{m=1}^{n-1}{\tau_i}^{-q_3}\prod_0(\tau_i=x_i)$
$\rho^{n-d_1(\tau_i)}I_1(\tau_i)/\sum_{m=1}^{n-1}\prod_0(\tau_i=x_i)\,\rho^{n-d_1(\tau_i)}I_1(\tau_i)]$(21)

V. NUMERICAL EXAMPLE

Let stress be a time function as given below and we assume that it suddenly changes at some unknown time τ , viz

$$S(t)=\begin{cases}2t & ;t<20\\ 2[20+(0.3)(t-20)] & ;t\geq 20, 0<\rho<1\end{cases}$$

We assume that the strength Y of component is a exponential random variable with mean strength $^1/_\theta$. Hence, life time X follows exponential distribution with distribution function

$F_x(t) = 1 - e^{-\alpha x}$;t<20

$= 1 - e^{-\alpha x'}$;t≥ 20

with α= 0.1, ρ= 0.3, x'=20+0.3(x-20). And we have generated 14 random observations from this model. As explained in section 6.3.α is considered a random observation from inverted gamma distributions with mean μ_0=0.1 and σ=0.1 resulting in a= 66 and b= 52.

Table 1: Generated observations from Model

i	1	2	3	4	5	6	7
Xi	15.77	8.88	6.4	3.55	2.2	22.81	16.27
i	8	9	10	11	12	13	14
Xi	5.22	20.1	52.59	188.21	50.38	78.83	174.09

We have calculated posterior mean, posterior mode and posterior median of τ under the informative and non-informative priors. These results are shown in Table2

Table 2: The Bayes estimates of change point τ for Model

Prior	Bayes estimates of change point τ		
	PosteriorMedian	PosteriorMean	PosteriorMode
Informative	19	20	22
Non-informative	18	18	23

We compute the Bayes estimates τ^*_L, τ^*_E, τ^{**}_L, τ^{**}_E of τ for the data given in Table-1 under the informative and non-informative priors using equations 18,19,20,21 respectively and results are shown in Table-3.

Table 3: The Bayes estimates using Asymmetric Loss Functions for Model

Shape parameter of asymmetric loss functions		Bayes estimates of change point with Informative prior		Bayes estimates of change point with Non-Informative prior	
q_1	q_3	τ^*_L	τ^*_E	τ^{**}_L	τ^{**}_E
0.09	0.09	20	20	18	19
0.10	0.10	19	20	18	19
0.20	0.20	20	20	18	19
1.2	1.2	19	18	18	17
1.5	1.5	18	17	17	18
-1.0	-1.0	21	22	22	23
-2.0	-2.0	22	23	23	24

Table 3 shows that for small values of |q|, q_1= 0.09, 0.1,0.2 Linex loss function is almost symmetric and nearly quadratic and the values of the bayes estimate under such a loss is not far from the posterior mean table 3 also shows that, for q_1=1.5, 1.2, Bayes estimate are less than actual value of τ = 20.
For q_1= q_3 = -1= -2, Bayes estimates are quite large than actual value τ = 20. It can be seen from table3 that the negative sign of shape parameter of loss function reflecting underestimation is more serious than overestimation. Thus problem of underestimation can be solved by taking the value of shape parameters of Linex and General Entropy loss function negative.
Table 3 shows that for small values of |q|, q_1= 0.09, 0.1,0.2 General Entropy loss function,the values of the bayes estimate under such a loss is not far from the posterior mean. Table 3 also shows that, for q_1=1.5, 1.2, Bayes estimate are less than actual value of τ = 20.
It can be seen Table 3 that positive sign of shape parameter of loss functions reflecting overestimation is more serious than under estimation. Thus problem of over estimation can be solved by taking the value of shape parameter of Linex and General Entropy loss function positive and high.

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REFERENCES

[1] Calabria, R. And Pulcini G.(1992):Bayes Probability intervals in a load –strength mobel. Commun,Statist-Ttheory meth, 21(12),3393-3405
[2] Calabria, R. And Pulcini G.(1994a):An engineering approach to Bayes estimation for the Weibull distribution, Microelectron Reliability, 34, 789-802.
[3] Bansal, A. K(1987b): Robustness of Bayes estimator for the mean of a normal population with non normal prior, Communication in statistics(Theory and methods), 7,453-600