Bayesian Estimation of Continuous Change Point in Exponential Distribution

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ABSTRACT
A patient is surviving according to exponential life time distribution. If at some unknown point of time \( \tau \) the Stress function changes then the life time distribution also changes. Exponential change point life time model is proposed. Bayes estimator of unknown change point \( \tau \) is obtained under asymmetric loss function.

Keywords—Variables, Function, GAMA

I. INTRODUCTION
A component fails when the stress induced by the operating conditions exceeds the stress resisting capacity (strength) of the component. We consider strength is random variables since it depend on several manufacturing variables such as temperature, size, surface finish etc. Now, if the strength changes with time then the reliability and hence the whole model is changed. Considering these possibilities, we proposed and study Bayesian estimation of exponential change point model. Mayuri Pandya(2004) had studied the Bayesian analysis of the inverse weibull change point model considering continuous change point in strength.

II. CHANGE POINT MODEL CONSIDERING CHANGE IN STRENGTH
Let, stress \( s(t) \) be a time increasing function which changes at some unknown time \( \tau \), viz
\[
S(t) = \begin{cases} 
\lambda t & ; t < \tau \\
\lambda (\tau + \rho(t - \tau)) & ; t \geq \tau 
\end{cases}
\]  
(1)
Let, we assume strength \( Y \) as an exponential random variable with mean value \( 1/\theta \), i.e the probability density function of \( Y \) is
\[
g(y) = \theta e^{-\theta y} ; \theta > 0, x > 0
\]  
(2)

Then the stress strength model is given by,
\[
R_x(t) = \text{Probability that component survive beyond time } t
\]
\[
= \int_{S(t)}^{\infty} g(y)dy = e^{-S(t)\theta}
\]  
(3)
So we propose following change point model (Stress change point model) related with

The stress–strength model is:
\[
R_x(t) = \begin{cases} 
e^{-\lambda \theta t} & ; t < \tau \\
e^{-\lambda (\tau + \rho(t - \tau))\theta} & ; t \geq \tau 
\end{cases}
\]
(4)
Hence,
\[
1 - F_x(t) = \begin{cases} 
e^{-\lambda \theta t} & ; t < \tau \\
K e^{-\lambda (\tau + \rho(t - \tau))\theta} & ; t \geq \tau
\end{cases}
\]
Where \( k \) is such that
\[
\int_{0}^{\infty} f(x)dx = 1
\]
Which gives
\[
k = 1
\]
Hence,
\[
F(X | \alpha, \beta) = \begin{cases} \alpha \exp[-\alpha x] & ; x < \tau \\
\rho \alpha \exp[-\alpha x] & ; x \geq \tau
\end{cases}
\]

Therefore, the Bayesian estimation of unknown change point \( \tau \) under asymmetric loss function is obtained.

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Where, 
\( x = 1 + p(x - \tau) \) \( \alpha = \lambda \theta \)

The likelihood function of \( \alpha \) and \( \tau \) given 
\( X = (X_1, X_2, \ldots, X_n) \) is

\[
L(\alpha, \tau | X) = \prod_{i=1}^{n} \{ae^{-ax_i}\}^{e_i} \{\rho ae^{-ax_i}\}^{1-e_i} 
\]

\[
= \rho^{n-d_1(\tau)} a^n \exp(-a \sum_{i=1}^{n} x_i e_i) \exp(-a \sum_{i=1}^{n} x_i (1-e_i)) 
\]

Where,
\( d_1(\tau) = \sum_{i=1}^{n} e_i = d_1(\tau, x) \)
\( A_1(\tau) = \sum_{i=1}^{n} x_i e_i = A_1(\tau, x) \)
\( A_2(\tau) = \sum_{i=1}^{n} x_i (1-e_i) = A_2(\tau, x) \)

\[
E_i = \begin{cases} 
1, & \text{if } x_i < \tau \\
0, & \text{if } x_i \geq \tau 
\end{cases} \quad (5)
\]

### III. POSTERIOR DENSITIES USING INFORMATIVE PRIOR IN MODEL-1 (INVERTED GAMMA PRIOR)

In this section, we derive marginal posterior density of \( \tau \) using informative prior. We suppose the marginal prior distributions of \( \alpha \) be inverted gamma distribution viz,

\[
g(\alpha) = b^\alpha \exp(-b/\alpha) / \Gamma(\alpha) 
\]

\( ; a = (\mu/\sigma)^2 + 2 \)
\( b = \mu(\mu/\sigma)^2 + 1 \) \( \quad (6) \)

If the prior information is given in terms of the prior mean and standard deviation then hyper parameters can be obtained by solving the equations,

\[
1 + \frac{2}{b} = \left[ 1 + \frac{1}{b} \right]^d 
\]

\[
\alpha = \frac{\ln(\mu)}{\ln(b/(b+1))} 
\]

For unknown change point \( \tau \), we assume that it takes one of the \( n \) observed values \( x_1, x_2, \ldots, x_n \) and taking discrete values with prior probability \( i = 1, 2, \ldots, n \).

Then the joint prior density of \( \alpha \) and \( \tau \), \( g_0(\alpha, \tau) \) is

\[
g_0(\alpha, \tau) = g_0(\alpha) \Pi_0(\tau = x_i) 
\]

\[
= b^\alpha \exp(-b/\alpha) \Pi_0(\tau = x_i) 
\]

\[
= K_j \alpha^{n-1} \exp(-b/\alpha) 
\]

(7)

Where,

\[
k_j = b^\alpha \Pi_0(\tau = x_i) 
\]

Joint posterior density of \( \alpha \) and \( \tau \), \( g_1(\alpha, \tau | x) \), is given by,

\[
g_1(\alpha, \tau | x) = L(\alpha, \tau | x) g(\alpha, \tau) / h_1(\chi) 
\]

\[
= \rho^{n-d_1(\tau)} a^n \exp(-a A_1(\tau, x) / h_1(\chi)) 
\]

\[
= \rho^{n-d_1(\tau)} b^\alpha \exp(-b/\alpha) \Pi_0(\tau = x_i) \exp(-a \sum_{i=1}^{n} x_i e_i) \exp(-a \sum_{i=1}^{n} x_i (1-e_i)) / h_1(\chi) 
\]

Where,

\[
h_1(\chi) = \sum_{i=1}^{n} \int_{0}^{\infty} \rho^{n-d_1(\tau)} b^\alpha \exp(-b/\alpha) \Pi_0(\tau = x_i) \exp(-a \sum_{i=1}^{n} x_i e_i) \exp(-a \sum_{i=1}^{n} x_i (1-e_i)) / h_1(\chi) 
\]

(8)

### USING NON-INFORMATIVE PRIOR

In this section, we derive marginal posterior density of \( \tau \) using non-informative prior.

Sometimes no prior information or technical knowledge about the parameters are available then we take Non-informative prior. Let us consider such non-informative prior densities on \( \alpha_1 \) and \( \alpha_2 \) to be,

\[
g_2(\alpha) = \frac{a}{\alpha} \quad \alpha > 0 
\]

\[
g_2(\alpha, \tau | x) = g_2(\alpha | x, \tau) \Pi_0(\tau = x_i) 
\]

\[
g_2(\alpha, \tau) = \frac{a}{\alpha} \Pi_0(\tau = x_i) 
\]

Now,

\[
L(\alpha, \tau | x) g_2(\alpha, \tau) = \left[ \frac{a}{\alpha} \Pi_0(\tau = x_i) \right] \rho^{n-d_1(\tau)} a^n \exp(-a \sum_{i=1}^{n} x_i e_i) \exp(-a \sum_{i=1}^{n} x_i (1-e_i)) 
\]

(12)

Joint posterior density of \( \alpha \) and \( \tau \), is given by,

\[
g_2(\alpha, \tau | x) = L(\alpha, \tau | x) g_2(\alpha, \tau) / h_2(\chi) 
\]

Where,

\[
h_2(\chi) = \sum_{i=1}^{n} \int_{0}^{\infty} L(\alpha, \tau | x) g_2(\alpha, \tau) d \alpha 
\]

(13)
\[
\begin{align*}
\frac{\sum_{i=1}^{n} \int \frac{n!}{i!} \rho^{i-1} \exp(-\alpha \rho) \exp(-\alpha \rho_{-i}(i-\epsilon)) \, d\alpha}{\int \rho^{n-d-1}(\rho) \, d\rho} = \\
\frac{\sum_{i=1}^{n} \int \frac{n!}{i!} \rho^{i-1} \exp(-\alpha \rho) \exp(-\alpha \rho_{-i}(i-\epsilon)) \, d\alpha}{\int \rho^{n-d-1}(\rho) \, d\rho}
\end{align*}
\]

Hence, marginal posterior density of change point \( \tau \),
\( g_2(\tau = x_i | x) \) is given by,
\( g_2(\tau = x_i | x) = \int_0^\infty g_2(\tau, x | x) \, d\tau \)

\[
\begin{align*}
g_2(\tau = x_i | x) &= \frac{1}{h_2(x)} \prod_{i=1}^{\infty} (\tau = x_i) \rho^{n-d-1}(\rho) \left\{ \int_0^\infty \exp(-\alpha (A_1(\tau) + A_2(\tau))) \, d\alpha \right\} \\
&= \frac{1}{h_2(x)} \prod_{i=1}^{\infty} (\tau = x_i) \rho^{n-d-1}(\rho) \left( \frac{\Gamma_1}{\alpha} \right) \\
&= \frac{1}{h_2(x)} \prod_{i=1}^{\infty} (\tau = x_i) \rho^{n-d-1}(\rho) \left( \frac{\Gamma_1}{\alpha} \right)
\end{align*}
\]

Where, \( h_2(x) = (A_1(\tau) + A_2(\tau))^n \)

IV. BAYES ESTIMATES

4.1 Using Informative Prior

In this section we obtain Bayes estimates of change point \( \tau \) using informative prior for this model. Expected loss function, \( E_1[L_1(\tau, x)] \) with respect to the posterior density, we get the Bayes estimate \( \tau^*_E \) of \( \tau \) using Linex loss function as,
\[
\tau^*_E = \frac{1}{q_1} \ln E_1[e^{-q_1}] = \frac{1}{q_1} \ln \left\{ \sum_{i=1}^{n} e^{-q_1} \rho^{n-d-1}(\rho) \left( \frac{\Gamma_1}{\alpha} \right) \right\}
\]

The joint posterior density of \( \alpha \) and \( \tau \) using informative prior for this model. Expected loss function as,
\[
\tau^*_E = \frac{1}{q_1} \ln E_1[e^{-q_1}] = \frac{1}{q_1} \ln \left\{ \sum_{i=1}^{n} e^{-q_1} \rho^{n-d-1}(\rho) \left( \frac{\Gamma_1}{\alpha} \right) \right\}
\]

Table 1: Generated observations from Model

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We have calculated posterior mean, posterior mode and posterior median of $\tau$ under the informative and non-informative priors. These results are shown in Table 2.

Table 2: The Bayes estimates of change point $\tau$ for Model

<table>
<thead>
<tr>
<th>Prior</th>
<th>Bayes estimates of change point $\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PosteriorMedian</td>
</tr>
<tr>
<td>Informative</td>
<td>19</td>
</tr>
<tr>
<td>Non-informative</td>
<td>18</td>
</tr>
</tbody>
</table>

We compute the Bayes estimates $\tau_L^*, \tau_E^*$, $\tau_L^{**}, \tau_E^{**}$ of $\tau$ for the data given in Table 1 under the informative and non-informative priors using equations 18,19,20,21 respectively and results are shown in Table 3.

Table 3: The Bayes estimates using Asymmetric Loss Functions for Model

<table>
<thead>
<tr>
<th>Shape parameter of asymmetric loss functions</th>
<th>Bayes estimates of change point with Informative prior</th>
<th>Bayes estimates of change point with Non-Informative prior</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1$</td>
<td>$q_3$</td>
<td>$\tau_L^*$</td>
</tr>
<tr>
<td>0.09</td>
<td>0.09</td>
<td>20</td>
</tr>
<tr>
<td>0.10</td>
<td>0.10</td>
<td>19</td>
</tr>
<tr>
<td>0.20</td>
<td>0.20</td>
<td>20</td>
</tr>
<tr>
<td>1.2</td>
<td>1.2</td>
<td>18</td>
</tr>
<tr>
<td>-1.0</td>
<td>-1.0</td>
<td>22</td>
</tr>
<tr>
<td>-2.0</td>
<td>-2.0</td>
<td>23</td>
</tr>
</tbody>
</table>

Table 3 shows that for small values of $|q|$, $q_1=0.09, 0.1, 0.2$ Linex loss function is almost symmetric and nearly quadratic and the values of the bayes estimate under such a loss is not far from the posterior mean table 3 also shows that, for $q_1=1.5, 1.2$, Bayes estimate are less than actual value of $\tau = 20$. For $q_1= q_3 = -1=-2$, Bayes estimates are quite large than actual value $\tau = 20$. It can be seen from table3 that the negative sign of shape parameter of loss function reflecting underestimation is more serious than overestimation. Thus problem of underestimation can be solved by taking the value of shape parameters of Linex and General Entropy loss function negative.

Table 3 shows that for small values of $|q|$, $q_1=0.09, 0.1, 0.2$ General Entropy loss function, the values of the bayes estimate under such a loss is not far from the posterior mean. Table 3 also shows that, for $q_1=1.5, 1.2$, Bayes estimate are less than actual value of $\tau = 20$. It can be seen Table 3 that positive sign of shape parameter of loss functions reflecting overestimation is more serious than underestimation. Thus problem of over estimation can be solved by taking the value of shape parameter of Linex and General Entropy loss function positive and high.

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REFERENCES