Development of Finite Element Models for Predicting Velocities and Pressure Distribution for Viscous Fluid Flow between two Parallel Co-Rotating Discs

J. A. Akpobi¹, SMG Akele²
¹Prof. in Production Engineering Department, University of Benin, Benin City
²Chief Lecturer in Mechanical Engineering Department, Auchi Polytechnic, Auchi

ABSTRACT

The study of flow of fluid between two parallel co-rotating discs continues to be of great concern to engineers and mathematicians. Previous works on this physical phenomenon have extensively applied experimental, analytical and numerical methods such as finite difference, spectral, and boundary element methods of analysis. In this paper we use the Galerkin-weighted residual finite element method to develop models for predicting velocities and pressure distribution for viscous-incompressible-steady-laminar-Newtonian fluid flow between two parallel discs co-rotating at the same angular velocity.

The models in this paper are developed by first reducing the non-linear Navier-Stokes equations to solvable non-dimensional governing differential equations. Galerkin-weighted residual finite element is then used to discretize the governing equations into element equations which are combined to form the domain equations. The domain equation is solved to obtain models to predict velocities and pressures distribution after duly imposing the boundary conditions. The resulting finite element solutions for radial velocity, tangential velocity and pressure are compared to their respective exact solutions.

The results show that as the number of elements is increased the better the finite element solution approximates the exact solution; and the more the radial and tangential velocity profiles assumed better parabolic profile. More so, the results obtained shows that finite element method solutions approximate well the analytical solutions, and the finite element results converge fast as the domain is refined.

Keywords--- Tesla pump, finite element, Navier-Stokes, viscous laminar flow

I. INTRODUCTION

Nikola Tesla in 1913 patented a turbine, referred to as bladeless turbine. This bladeless turbine is based on the principle of centripetal fluid flow over a bladeless disc surface; with the fluid and disc interface being governed by the adhesion and boundary layer effects. (http://en.wikipedia.org/wiki/Tesla_turbine). This implies that unlike the conventional turbine, the fluid does not impinge on blades or vanes.

Flow of fluid in Tesla turbomachinery can be categorized into two main configurations: turbine and pump configuration, which are usually simplified into radial inflow or outflow, swirling or non-swirling flow, between two closely-spaced parallel co-rotating discs by researchers.

The continuing need for pump performance and efficiency in recent years has brought to the fore the need for more accurate knowledge of flow conditions between two co-rotating parallel discs in discs pumps. Fluid flow between closely spaced parallel discs problem has been of great concern as a result of its specific uses in devices such as turbomachinery and micropolar pumps. (Thakhar et al, 2000). Consequently, a lot of experimental and theoretical studies have and are being carried out on fluid flow between closely-spaced parallel discs systems. Most of these studies have been carried by researchers and/or organizations resulting in the development of various analytical and numerical formulations/models to predict the nature of flow fields between parallel discs, especially relating to Tesla turbine or pump.

According to Batista (2007), the flow of fluid between closely spaced discs was studied by Breiter and Pohlhausen (1962). From the NS equations they derived formulations for velocity components and pressure; their results show that the solution depends importantly on discs spacing and angular velocity. Batista (2007) continued that Breiter and Pohlhausen (1962) used the finite difference method with constant inlet profile to provide solution to the non-linear equations. The work of Breiter and Pohlhausen was continued by Rice and co-workers, who used different methods to obtain velocity and pressure distribution in the flow of fluid between two parallel discs. Then Boyd and Rice (1968) used the finite difference method to obtain solution for velocity and pressure for different inlet velocities in; while
Boyaack and Rice (1971) used the integral method to analyze flow between two discs. Ho-Yan (2011) reported that Lawn and Rice (1974) are known to have employed findings from experimental and analytical methods, which include the analyses of laminar flow between two co-rotating discs (Boyd and Rice, 1968) to provide data for the design and performance of a Tesla turbine based on volumetric flow rate, Reynolds number and radii ratio. Sim and Yang (1984) noted that rotating mechanical devices associated with heat and mass transfer are abundant in industry. For example as: turbomachinery, rotating heat exchanger, rotating-disc contractors. More so, rotating-disc systems can be into free disc, rotor-stator, and rotating cavities systems.

Most studies (Tsifourardis, 2003; Batista, 2007; Ghaly and Vatistas, 2001; Oliveira and Pascao, 2009; Yu et al, 2012; Matej, 2011; Sengupta and Guha, 2012; Takhar et al., 2000; Torii and Yang, 2008) we reviewed have been carried out based on experimental and/or numerical pure Poiseuille-Couette through flow (inflow and/or outflow).

The main equations governing viscous fluid flows are the continuity and Navier-Stokes (NS) equations. Navier-Stokes and continuity equations form a set of coupled non-linear partial differential equations that are difficult to apply directly in flow analysis, especially to flow domain with irregular geometries and boundary conditions. Intrinsically, Navier-Stokes equations are non-linear as a result of the convective terms they contain. The non-linearity of Navier-Stokes equations makes it essentially difficult to solve analytically or numerically directly without one or two transformations and imposition of appropriate assumptions. (Dechaumphai and Kanjanakljkasem, 1999; Oliveira and Pascao, 2009; Dong et al, 2008; Sengupta and Guha, 2012; Batista, 2007; Ghaly and Vatistas, 2001s).

The solutions of NS equations by analytical manipulations are often a formidable task and difficult because of the inherent constraints in the NS equations, as a result, methods such finite difference and Rayleigh-Ritz variational approaches are mostly employed. (Reddy, 1993). Though, Dechaumphai and Kanjanakljkasem (1999) noted that in the past, the finite difference method was the most popular method used to provide accurate solutions to flow problems, but the method was not suitable when it comes to irregular domains, especially in the regions with complex flow characteristics. That in spite of finite element method application in many engineering fields, its application in fluid dynamics is still under study because of the inherent non-linearity of the NS equations governing fluid flows. Therefore, the finite element method is better employed to fluid flow problems because of its ability to discretize a given flow domain into sub-domains.

This paper is on the use of Galerkin-weighted-residual finite element method to develop models for predicting velocities and pressure distribution in two-dimensional, viscous-incompressible-steady-laminar-Newtonian fluid flowing between two parallel discs annulus that are co-rotating at the same constant angular velocity in the same direction.

**II. MODEL GEOMETRY**

Consider model geometry for two dimensional cylindrical coordinates system which coincides with shaft axis. That is, the domain we considered for finite element analysis is a 2D space between two co-rotating discs that are separated along the z-axis by distance 2b (the disc- gap), figure 1 above. The flow field is assume to be an axisymmetric (axially symmetric) with the origin taken mid-way (centreline) between the discs. Let 2b (figure 2 below) be the disc-gap, distance between the discs. In cylindrical coordinates system (r, θ, z), the faces of the two discs are given by z = -b (disc 1) and z =+b (disc 2). Let u, v, w and the set $\omega_1$, $\omega_2$, $\omega_3$ be the velocity components and angular velocity components in the cylindrical coordinate system in r, θ, z directions respectively.
Where, \( \kappa \) and \( \eta \) represent non-dimensional radii ratio and axial axis respectively.

### III. MATHEMATICAL FORMULATIONS

1. Continuity equation (CE)

\[
\frac{1}{r^*} \frac{\partial}{\partial r^*} \left( r^* u_r \right) + \frac{1}{r^* \frac{\partial}{\partial \theta^*}} \frac{\partial v_r}{\partial z^*} + \frac{\partial w_z}{\partial z^*} = 0
\]

2. \( r \)-momentum equation (r-ME)

\[
\frac{u_r}{r^* \frac{\partial}{\partial r^*}} + \frac{v_r}{r^* \frac{\partial}{\partial \theta^*}} - \frac{v_r^2}{r^*} + \frac{w_z}{r^* \frac{\partial}{\partial z^*}} = -\frac{1}{\rho^*} \frac{\partial p^*}{\partial r^*} + \nu^* \left[ \frac{1}{r^* \frac{\partial}{\partial r^*}} \left( \frac{\partial}{\partial r^*} \left( r^* u_r \right) \right) + \frac{1}{r^* \frac{\partial}{\partial \theta^*}} \left( \frac{\partial}{\partial \theta^*} \left( r^* v_r \right) \right) - \frac{2}{r^*} \frac{\partial v_r}{\partial \theta^*} + \frac{1}{r^* \frac{\partial}{\partial z^*}} \left( \frac{\partial}{\partial z^*} \left( u_r \right) \right) \right] + \rho g_r
\]

3. \( \theta \)-momentum equation (\( \theta \)-ME)

\[
\frac{u_r}{r^* \frac{\partial}{\partial r^*}} + \frac{v_r}{r^* \frac{\partial}{\partial \theta^*}} - \frac{u_r v_r}{r^*} + \frac{w_z}{r^* \frac{\partial}{\partial z^*}} = -\frac{1}{\rho^*} \frac{\partial p^*}{\partial \theta^*} + \nu^* \left[ \frac{1}{r^* \frac{\partial}{\partial r^*}} \left( \frac{\partial}{\partial r^*} \left( r^* v_r \right) \right) + \frac{1}{r^* \frac{\partial}{\partial \theta^*}} \left( \frac{\partial}{\partial \theta^*} \left( r^* v_r \right) \right) + \frac{2}{r^* \frac{\partial}{\partial \theta^*}} \frac{\partial v_r}{\partial \theta^*} + \frac{1}{r^* \frac{\partial}{\partial z^*}} \left( \frac{\partial}{\partial z^*} \left( v_r \right) \right) \right] + \rho g_\theta
\]

4. \( z \)-momentum equation (\( z \)-ME)
In order to simplify the non-linear Navier-Stokes equations above, we applied the following assumptions and boundary conditions below.

**Assumptions:**

(i) that flow is steady (i.e. \( u^*, v^*, w^* \) and \( p^* \) are independent of time)

(ii) that flow is in radial and tangential directions only

(iii) axisymmetric flow implies that \( u^*, v^* \) and \( p^* \) do not depend on \( \theta \)

(iv) fully-developed flow implies that \( u \) does not depend on \( r^* \)

(v) that fluid is isotropic with constant properties,

(vi) that body forces are negligible,

(vii) that no-slip condition exist at disk face (i.e. \( u (-b) = u (+b) = 0 \)),

(viii) the vena contracta effect is neglected,

(ix) that analysis are carried out with discs angular velocities equal \( \omega_{d1} = \omega_{d2} = \omega \).

**Boundary conditions,**

\[
\begin{align*}
\frac{\partial}{\partial r^*} \left( r^* u^*_r \right) = 0,
\frac{\partial}{\partial r^*} \left( r^* u^*_\theta \right) = 0,
\frac{\partial}{\partial r^*} \left( r^* u^*_z \right) = 0,
\frac{\partial}{\partial \theta^*} \left( \theta^* u^*_r \right) = 0,
\frac{\partial}{\partial \theta^*} \left( \theta^* u^*_\theta \right) = 0,
\frac{\partial}{\partial \theta^*} \left( \theta^* u^*_z \right) = 0,
\frac{\partial}{\partial z^*} \left( \theta^* u^*_r \right) = 0,
\frac{\partial}{\partial z^*} \left( \theta^* u^*_\theta \right) = 0,
\frac{\partial}{\partial z^*} \left( \theta^* u^*_z \right) = 0
\end{align*}
\]

By imposing the above assumptions and boundary conditions on the continuity equation (1), \( r^* \)-, \( \theta^* \)- and \( z^* \)-momentum equations (1), (2), (3) and (4), the equations are simplified into a 2D problem below.

1. Continuity equation (CE)

\[
\frac{\partial}{\partial r^*} \left( r^* u^*_r \right) = 0
\]

2. \( r^* \)-momentum equation (r-ME)

\[
\frac{\partial}{\partial r^*} \left( r^* u^*_r \right) - \frac{v^2}{r^2} = - \frac{1}{\rho^*} \frac{\partial p^*}{\partial r^*} + v^* \left[ \frac{1}{r^*} \frac{\partial^2 (u_r)}{\partial z^*} \right]
\]

3. \( \theta^* \)-momentum equation (\( \theta^* \)-ME)

\[
\frac{\partial}{\partial \theta^*} \left( \theta^* u^*_r \right) + \frac{1}{r^*} \frac{\partial}{\partial r^*} \left( r^* u^*_\theta \right) = - \frac{1}{\rho^*} \frac{\partial p^*}{\partial \theta^*} + v^* \left[ \frac{1}{r^*} \frac{\partial^2 (u_\theta)}{\partial z^*} \right]
\]
4. z-momentum equation (z-ME)

\[ 0 = - \frac{1}{\rho^*} \frac{\partial p^*}{\partial z^*} \]

Equation 10 is an expression for Coriolis effects which relate the absolute tangential velocity component \((v_t)\) of disc to inertial relative tangential velocity \((v_\theta)\) of the fluid and the mean velocity \((\bar{v})\) in the rotating fluid as follows (Allen, 1990):

\[ v_t = v_\theta + \bar{v} = v_\theta + r\omega \]
\[ v_\theta = v_t - r\omega \]

Also, the dimensional variables in equations 6, 7, 8 and 9 can be expressed in non-dimensional form as (Batista, 2007; Lawn and Rice, 1974):

\[
\begin{align*}
U_o &= \frac{U_o^*}{\omega^* R_o^*}, V_o = \frac{V_o^*}{\omega^* R_o^*}, \xi = \frac{b^*}{R_o^*}, \\
Re &= \frac{\rho^* \omega^* b^{*2}}{\mu^*}, Re_r = \xi Re, \alpha = \frac{V_o^*}{U_o^*}
\end{align*}
\]

Where \(u, v, p, U_o, V_o, r, z, b, R_o\), and \(p_o\) are dimensional parameters while \(a, b, \eta, \kappa, u, v, p, U_o, \) and \(V_o\), \(Re, a, b, \eta, \kappa\) (<<1 aspect ratio) are the dimensionless parameters. Note that \(U_o, V_o\) and \(p_o\) are the dimensionless mean radial velocity, mean tangential velocity and mean pressure respectively.

By substituting the Coriolis effect and the non-dimensional parameters above into equations 6, 7, and 8, with the quadratic terms in \(v_\theta\), \(u\) and \(\omega\) are ignored (Podergajs, 2011) the dimensional equations 6, 7, and 8 reduce to the following non-dimensional forms,
IV. METHOD OF SOLUTION

Galerkin-weighted residual finite element method for linear rectangular elements (Reddy, 1993) is employed to provide solution.

We first discretized the domain $\Omega$ (0 ≤ $\kappa$ ≤ 0.1; -1 ≤ $\eta$ ≤ +1) into 4 rectangular elements mesh with nine nodes, along the $\kappa$- and $\eta$-axes respectively (see figure 4 below) with a typical element having interval of ‘a’ (= $\kappa_{i+1}$ - $\kappa_i$) along the $\kappa$-axis and b (= $\eta_{i+1}$ - $\eta_i$) along the $\eta$-axis. Here a = ($\kappa_o$ - 0)/2 = $\kappa_o$/2 and b = $2\eta$/2.

In the following sessions, we use the Galerkin-weighted residual finite element method (Reddy, 1993) to solve the derived non-dimensional governing differential equations and their boundary conditions.

(1) Determination of radial velocity ($r$-momentum)

The differential equation (18) below is the derived governing equation for radial velocity $u$ subject to the indicated boundary conditions.

\[
-2\alpha^2 V \omega \kappa = -\kappa \frac{dp}{d\kappa} + \frac{1}{Re_r} \left( \frac{\partial^2 u}{\partial \eta^2} \right)
\]  

Subject to these boundary conditions:

\[
\kappa = \kappa_o : u_{\kappa_o(-\eta)} = 0, U_{(CL)} = U_{\max}, u_{\kappa_o(+\eta)} = 0
\]

\[
\kappa = \kappa_i : u_{\kappa_i(-\eta)} = 0, \frac{dU_{\max}}{d\eta} = 0, u_{\kappa_i(+\eta)} = 0
\]
The Galerkin scheme is therefore given as
\[
\int_{\Omega} w R d\Omega = 0
\]
Where the Galerkin-weighted residual integral is
\[
\int_{\Omega} w \left[ -\frac{1}{\text{Re}} \frac{d^2 u}{d\eta^2} + \kappa \frac{dp}{d\kappa} - 2\omega \alpha^2 V \kappa \right] d\Omega = 0
\]
The weak form of this equation is
\[
0 = \int_{\Omega} w \left[ -\frac{1}{\text{Re}} \frac{d^2 u}{d\eta^2} + \kappa \frac{dp}{d\kappa} - 2\omega \alpha^2 V \kappa \right] d\Omega
\]
\[
= \int_{\Omega} \left[ \frac{1}{\text{Re}} \frac{dw}{d\eta} \frac{du}{d\eta} + wK \frac{dp}{d\kappa} - 2w0 \alpha^2 V \kappa \right] d\Omega - \frac{w}{\text{Re}} \frac{du}{d\eta} |_{\Gamma}
\]
Where,
\(\Omega = \text{domain of fluid body between discs} = 2\pi d R d\eta d\theta\)
\(w = \text{weight function which depends on } \kappa \text{ and } \eta.\)

Let \(u(\kappa, \eta)\) be approximated by the finite element interpolation \(u^e\) over the element domain \(\Omega^e\), then,
\[
u(\kappa, \eta) = u^e(\kappa, \eta) = \sum_{j=1}^{n} u^e_j(\kappa, \eta)
\]
\[
\frac{du}{d\eta} = \sum_{j=1}^{n} u^e_j(\eta) \frac{d\psi_j}{d\kappa}
\]
Also let \(w\) the weight function be
\[
w(\kappa, \eta) = \psi_i(\kappa, \eta)
\]
So that
\[
\frac{dw}{d\kappa} = \frac{d\psi_i}{d\kappa}
\]
\[
\frac{dw}{d\eta} = \frac{d\psi_i}{d\eta}
\]
Where,
i, j = 1,2,3,...n,
\(\psi_j(\kappa, \eta)\) = the jth interpolation functions of element e

By substituting equations 27 through 29 into 26 yields,
\[
0 = \int \left[ \frac{1}{Re_r} \frac{dw}{d\eta} \frac{du}{d\eta} + wk \frac{dp}{d\kappa} - 2w_\omega \alpha V \kappa \right] d\Omega - \frac{w}{Re_r} \frac{du}{d\eta} \bigg|_r \\
= \int \int \int \left[ \frac{1}{Re_r} \frac{dw}{d\eta} \frac{du}{d\eta} + wk \frac{dp}{d\kappa} - 2w_\omega \alpha V \kappa \right] \cdot 2\pi \kappa \kappa \kappa \kappa d\kappa d\eta d\theta - \frac{2\pi w}{Re_r} \frac{du}{d\eta} \bigg|_r \\
= \int \int \int \left[ \frac{1}{Re_r} \frac{dw}{d\eta} \sum_{j=1}^{n} u_j \frac{dy_j}{d\eta} - \kappa \psi, q_o - 2\psi, \alpha \omega V \kappa \right] \cdot 2\pi \kappa \kappa \kappa \kappa d\kappa d\eta d\theta - \frac{2\pi \psi, i}{Re_r} \frac{du}{d\eta} \bigg|_r \\
= 2\pi \sum_{j=1}^{n} U_j \int \int \left[ \frac{1}{Re_r} \kappa \psi, q_o - 2\kappa^2 \psi, \alpha \omega V \kappa \right] d\kappa d\eta - \int \int \left[ \kappa^2 \psi, q_o + 2\kappa^2 \psi, \alpha \omega V \kappa \right] d\kappa d\eta - \frac{2\pi q_o}{Re_r} \int \kappa \psi, d \\
= \sum_{j=1}^{n} \left[ M^e_{ij} U^e_j \right] = \left\{ f^e_i \right\} - \left\{ Q^e_i \right\}
\]

Where

\[
q_n = \left[ \frac{du}{d\eta} - \frac{du}{d\eta} \right]_n = \frac{du}{d\eta}_n
\]

\[
q_o = - \frac{dp}{d\kappa}
\]

\[
\Gamma = \text{elementsurface}
\]

In matrix form

\[
\left[ M^e_{ij} \right] U^e_j = \left\{ f^e_i \right\} + \left\{ Q^e_i \right\}
\]

Where,

\[
M^e_{ij} = 2\pi \sum_{j=1}^{n} U_j \int \int \left[ \frac{1}{Re_r} \kappa \psi, q_o - 2\kappa^2 \psi, \alpha \omega V \kappa \right] d\kappa d\eta
\]

\[
f^e_i = \int \int \left[ \kappa^2 \psi, q_o + 2\kappa^2 \psi, \alpha \omega V \kappa \right] d\kappa d\eta
\]

\[
Q^e_i = \frac{2\pi q_o}{Re_r} \int \kappa \psi, d
\]

The assembled element equations for the domain of four elements mesh with nine nodes in matrix form is
Imposition of the following boundary conditions, no slip condition at disc face,

\[
\kappa = \kappa_o : u_{\kappa o(-\eta)} = 0, U_{\omega(CL)} = U_{\max}, u_{\kappa o(+\eta)} = 0
\]

\[
\kappa = \kappa_i : u_{\kappa i(-\eta)} = 0, \frac{dU_{\max}}{d\eta} = 0, u_{\kappa i(+\eta)} = 0
\]

Finite element solution function is

\[
u^\alpha(\eta) = \frac{Re_r \cdot b \left(q_o + 2 \alpha^2 \omega V\right)}{2a^3} \kappa \left(1 - \eta^2\right)
\]

\[
u^\beta(\kappa) = \frac{Re_r \cdot b \left(q_o + 2 \alpha^2 \omega V\right)}{2a^3} \kappa; At: \eta = 0(centrelin)
\]

The finite element analysis steps (1) to (5) above are similarly used to determine tangential velocity and pressure solution in the next sections.

(2) Determination of tangential velocity (\(\theta\)-momentum)

\(\theta\)-momentum equation in the dimensionless form is analyze here to determine tangential velocity component from the equation governing below,

\[
\theta - \text{momentum}:
\]

\[
2U\alpha^2 \omega \kappa = \frac{1}{Re_r} \left(\frac{\partial^2 \nu}{\partial \eta^2}\right)
\]

Subject to boundary conditions,

\[
\kappa = \kappa_o : \nu_{\kappa o(-\eta)} = \kappa_o \omega, V_{\omega(CL)} = V_{\max}, \nu_{\kappa o(+\eta)} = \kappa_o \omega
\]

\[
\kappa = \kappa_i : \nu_{\kappa i(-\eta)} = 0, \frac{dV_{\max}}{d\eta} = 0, \nu_{\kappa i(+\eta)} = 0
\]

Finite element solution functions are,
Pressure model is determined from the governing equation used for radial velocity. The governing equation is

\[ \frac{\alpha \omega \kappa}{a^2} \frac{d}{d\kappa} \left( \frac{1 - \eta^2}{1} \right) \]

Subject to following boundary conditions,

\[ p_1 = p_5 = p_6 = p_7 = p_9 = 0 \]

\[ p_{\kappa(\eta=0)} = 0 \]

Finite element solutions function is

\[ \frac{\alpha \omega \kappa}{a^2} \left( \kappa - \kappa \eta^2 \right) \]

V. RESULTS AND DISCUSSION

The results reported here first compare analytical solutions with FE solutions. 

Radial velocity: Finite Element Solutions against Analytical Solutions

In figures 4 and 5 below, radial velocity versus disc-gap graphs show that both FEM and analytical solution satisfy the no-slip boundary condition at discs walls; and that radial velocity profile in the flow domain are parabolic for both FE solution and Exact solution, which is an indication of viscous laminar flow.

Figure 6 compares exact solution with FE solutions for 4, 16 and 32 elements. This graph shows that the analytical solution is approximated as the domain is refined (i.e. number of elements is increased). For \( \text{Re} = 0.9, 5, 15 \) and \( \alpha = 0.2, \omega = 52.36, \kappa = 1, q_o = 1800 \)

In figure 7 is a graph of radial velocity versus radii ratio along the centre line. The curve shows that radial velocity increases with radii ratio along half disc gap in the flow domain.

Tangential velocity: Finite Element Solutions against Analytical Solutions for Tangential velocity against Radii Ratio

In figures 8 and 9, tangential velocity is plotted against disc-gap. The graphs show that for both FE and exact solution the angular velocities at discs inlet are zero, which satisfy the assumption made (\( \omega_1 = \omega_2 = 0 \)). The curves also show that tangential velocity increases in the direction of flow (i.e. in the direction of increasing radius) with parabolic profile.

Figure 10 below compares analytical solution (red curve) with FE solution (blue curve) for tangential velocity \( v \) against discs gap \( \eta \). From the figure it is shown that the FE solutions approximate fast the exact solution. For number of elements \( N = 4 \) (\( a = 0.05, b = 0.005 \)), \( 16 \) (\( a = 0.025, b = 0.0025 \)), and \( 32 \) (\( a = 0.0125, b = 0.00125 \)); \( \text{Re}_{t} = 0.2, \alpha = 0.2, \omega = 52.36, \kappa = 1, q_o = 1800 \)

Tangential velocity: Finite Element Solutions against Analytical Solutions for Tangential velocity against Radii Ratio

Figures 11 and 12 show plots of tangential velocity versus radii ratio long the mid-section of the domain for FEM and analytical solution. the analytical solution exhibit the effect of no-swirl action by the curve line unlike the FEM plot. However, both show that tangential velocity increase with radii as is expected of a pump configuration.

Figure 13 compares analytical and FE solutions for tangential velocity (\( v \)) against radii ratio \( \kappa \). The red curve depicts analytical curve. The graph shows that FE solutions approximate well the exact solution. For number of elements \( N = 4 \) (\( a = 0.05, b = 0.005 \)), \( 16 \) (\( a = 0.025, b = 0.0025 \)), and \( 32 \) (\( a = 0.0125, b = 0.00125 \)); \( \text{Re}_{t} = 0.2, \alpha = 0.2, \omega = 52.36, \kappa = 1, q_o = 1800 \)

Pressure: Finite Element Solutions against Analytical Solutions

Figures 14, 15 and 16 all show that pressure varies along the disc gap reaching maximum value at the disc gap mid-section, and that it depends on radii ratio too.
Figure 17 below compares analytical and FE solutions for pressure (p) against radii ratio κ. The red curve depicts analytical curve while the blue represents FE curve.

(1) Finite element method: Plot of $u(\eta)$ vs disc-gap $\eta$ ($= 2b$)

\[
\frac{\text{radial velocity, } u}{Rer \cdot \kappa \cdot b \cdot \left(q_0 + 2\alpha^2 \cdot \omega \cdot V\right)} \left(1 - \eta^2\right) \approx 2 \cdot a^3
\]

![Fig. 4: Radial velocity-discs gap profile](image)

(2) Analytical solution: radial velocity $u$ against distance between discs.

\[
\left(1 - \eta^2\right) \left(\alpha^2 \cdot V \cdot \omega + \frac{q_0}{2}\right) \cdot \text{Re}_r 100
\]

![Fig. 5: Radial velocity vs discs-gap](image)
(3) Comparison of finite element with exact solution

\[ U = \frac{(1-\eta^2)(\alpha^2 \cdot \nu \cdot \omega + \frac{q_0}{2}) \cdot Re \cdot 1000}{2 \cdot 0.053^3} \]

\[ Re \cdot \frac{\kappa \cdot 0.005 (q_0 + 2\alpha^2 \cdot \nu \cdot \omega)}{2 \cdot 0.025^3} (1-\eta^2) \]

\[ Re \cdot \frac{\kappa \cdot 0.0025 (q_0 + 2\alpha^2 \cdot \nu \cdot \omega)}{2 \cdot 0.0125^3} (1-\eta^2) \]

Legend

- Red: Analytical solution
- Blue: Finite element solution (4 elements)
- Green: Finite element solution (16 elements)
- Pink: Finite element solution (32 elements)

Fig. 6: Analytical solution against FE solutions

(4) Finite element method: Plot of radial velocity (u) against radii ratio \( \kappa \) (centre line value)

\[ U = \frac{Rer \cdot b \cdot (q_0 + 2\alpha^2 \cdot \omega \cdot \nu)}{2 \cdot a^3} \cdot \kappa \]

Fig. 7: Radial velocity-radii ratio profile
Finite element: Plot of tangential velocity, \( v(\eta) \) vs. disc-gap, \( \eta \).

![Finite element plot](image)

Fig. 8: Tangential velocity-discs gap profile

Analytical solution: tangential velocity vs. against distance between discs.

![Analytical solution plot](image)

Fig. 9: Tangential velocity-discs gap profile

Comparison of finite element with analytical solution.

![Comparison plot](image)
Legend

Red  Analytical solution
Blue  Finite element solution (4 elements)
Green Finite element solution (16 elements)
Pink  Finite element solution (32 elements)

Fig. 10: Analytical solution vs FE solution

(8)  Finite element: Plot of tangential velocity (v) against radii ratio κ at centreline, at \( \eta = 0 \)

![Graph](https://example.com/graph1.png)

\[
\frac{Rr \cdot b \cdot \alpha^2 \cdot \omega \cdot U}{a^3}(\kappa)
\]

Fig. 11: Tangential velocity-radii ratio profile

(9)  Analytical solution: Tangential velocity (v) against radii ratio κ

![Graph](https://example.com/graph2.png)

\[
\omega \cdot (\kappa^2 + \kappa) + \kappa \cdot V
\]

Fig. 12: Tangential velocity-radii ratio profile
(10) Comparison of finite element with analytical solution

\[ \omega \left( \kappa^2 + \kappa \right) + \kappa \cdot V \]
\[ \text{Re} \cdot 0.005 \cdot \alpha^2 \cdot \omega \cdot U \kappa \]
\[ 0.05^3 \]
\[ \text{Re} \cdot 0.0025 \cdot \alpha^2 \cdot \omega \cdot U \kappa \]
\[ 0.025^3 \]
\[ \text{Re} \cdot 0.00125 \cdot \alpha^2 \cdot \omega \cdot U \kappa \]
\[ 0.0125^3 \]

![Graph showing tangential velocity vs radii ratio](image)

**Legend**

<table>
<thead>
<tr>
<th>Color</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td>Analytical solution</td>
</tr>
<tr>
<td>Blue</td>
<td>Finite element solution (4 elements)</td>
</tr>
<tr>
<td>Green</td>
<td>Finite element solution (16 elements)</td>
</tr>
<tr>
<td>Pink</td>
<td>Finite element solution (32 elements)</td>
</tr>
</tbody>
</table>

Fig. 13: Tangential velocity-radii ratio profiles

(11) Finite element: Plot of pressure \( p (\eta) \) vs discs gap \( \eta \).

\[ \frac{\alpha^2 \cdot V \cdot \omega \cdot b \cdot \kappa}{a^3} (1 - \eta^2) \]

![Graph showing pressure vs discs gap](image)

Fig. 14: Pressure-discs gap profile
Finite element: Plot of pressure $p(\kappa)$ vs radii ratio $\kappa$ at centreline ($\eta = 0$)

Analytical solution: Pressure ($p$) against radii ratio $\kappa$
VI. CONCLUSION

Following the study, we therefore draw the following conclusions:

Since the NS equations for fluid flow has the inherent difficult of being solved directly analytically, certain assumptions were assumed to simplify the NS equations into three asymptotic close-form analytical governing equations for radial and tangential velocity as well as for pressure distribution.

The developed three governing equations were discretized using Galerkin-weighted residual finite element method. The solutions obtained by FEM solutions are compared with the exact solutions.

Lastly, finite element method was found to be a good approximation to exact solutions. We observed that the results obtained are highly accurate and converge well with the exact solutions as the domain is refined (that is, as the number of elements is increased the more the FE solution approximates the exact solution and the more the velocity profile assumed better laminar profile).

RECOMMENDATIONS

1. The finite element method has not been fully applied to fluid flow between parallel co-rotating discs. Hence it is recommended that other element shapes and sizes be analyzed.
2. FE procedure used in this present work should be applied to turbine configuration involving heat transfer.
REFERENCES


