Evaluation and Estimation of Reliability Characteristics of some System Configurations with Linearly Increasing Hazard Rates and Type II Censored Samples

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ABSTRACT

Estimation of the parameters and other reliability characteristics based on complete sample leads to excess wastage of cost and time. Whenever the high cost products are under study, it is better to use number of failure-censored samples (Type II censored). Following the concept, the purpose of the present study is to highlight an alternative easier method of construction of structure function of complex systems and evaluation. Further, the estimation of the involved parameter and different reliability characteristics with type II censored sampling has been carried out.

Keywords-- Type II censoring, Structure Function, Reliability functions, MTSF and hazard rate functions

I. INTRODUCTION

Reliability analysis of a complex system is mainly based on the following two steps:
1. Construction of the structure function and hence find out the expressions for some important reliability characteristics under some given specific assumptions.
2. Estimation of the parameters of the basic life time distribution(s) and other reliability characteristics of the components of the system as well as for the whole system also.

Now a days, the demand of multi-objectively of the products leads to the construction of more and more complex products. For example, the extra functions of radio, camera, video, music system, clock etc are expected to be in a mobile phone which leads the complexity in the mobile phone system. The construction of structure function becomes cumbersome as the complexity of the system increases. In this situation, an easier method for the construction of the structure function of such type complex system is described.

Estimation of the parameters and other reliability characteristics based on complete sample leads to excess wastage of cost and time. Whenever the high cost products are under study, it is better to use number of failure-censored samples (Type II censored).

Keeping the above facts in view, the purpose of the present chapter is to highlight an alternative easier method of construction of structure function of complex systems and evaluation as well as estimation of the involved parameter and different reliability characteristics with type II censored sampling has been carried out. In the present study, the failure rates of the components are taken linearly increasing, i.e. life time distribution will be Rayleigh.

II. RAYLEIGH DISTRIBUTION

Dyer and Whisened (1973) found Rayleigh distribution as extremely important in communication engineering. Siddiqui (1962) used Rayleigh distribution as radio wave power distribution while Polovko (1960) found it suitable for electro vacuum devices. Rayleigh distribution is especially suitable for the life testing of components or products that age with time i.e. the failure rate increases as time increases.

The Rayleigh probability density function (p.d.f.) of a random variable X is given by

\[ f(x, k) = kx e^{-kx^2/2} \quad ; \quad x, k > 0 \quad ...(1) \]
The exponential distribution widely used in reliability studies holds for the time to failure of a component with constant failure rate, while Rayleigh distribution holds for a component with linearly increasing failure rate. The density function $f(x)$ is sketched in Fig. 1

\[ f(x) = \frac{x}{\sqrt{2\pi k}} e^{-x^2/(2k)} \quad \text{for} \quad x > 0 \]

It should be noted that

\[ P \left[ X \geq \sqrt{2/k} \right] = e^{-1} \quad \text{and} \quad P \left[ 0 \leq X \leq \sqrt{2/k} \right] = e^{-1} \]

\[ E(X) = \sqrt{\frac{\pi}{2k}} , \quad V(X) = \frac{2}{k} \left(1 - \frac{\pi^2}{4}\right) \quad \text{and} \]

\[ \text{C.V.} = \sqrt{\frac{V(X)}{E(X)}} = \sqrt{\frac{4 - \pi^2}{\pi}} , \quad \text{(independent of} \quad k) \]

(3) The cumulative distribution function of Rayleigh distribution is given by

\[ F(X) = P \left[ X \leq x \right] = \begin{cases} 1 - e^{-kx^2/2} & \text{for} \quad x > 0 \\ 0 & \text{otherwise} \end{cases} \]

Hence the reliability function for the mission time

't' becomes as

\[ R(t) = \begin{cases} 1 - e^{-kt^2/2} & \text{for} \quad t > 0 \\ 0 & \text{otherwise} \end{cases} \]

and

\[ \text{MTSF} = \sqrt{\frac{\pi}{2k}} \]

Also, the failure rate is given by

\[ h(t) = \frac{f(t)}{R(t)} = kt \quad ; \quad k, \ t > 0 \]
It is clear that \( h(t) \) is linearly increasing function of time (Fig. 2).

![Graph showing \( h(t) \) as a linearly increasing function with \( k \) as a parameter](image)

**III. TOTAL TIME ON TEST (TTT) TRANSFORMATION**

The total time on test transformation of the distribution \( F \) is given by

\[
H_F^{-1} (v) = \int_0^v \left[ 1 - F(u) \right] du ; \quad 0 \leq v \leq 1
\]

...(2)

It can be shown under assumption of general nature that there is one to one correspondence between \( F(t) \) and \( H_F^{-1} (v) \). The scaled TTT transform of \( F(t) \) is defined as

\[
\varphi_F(v) = \frac{H_F^{-1}(v)}{H_F^{-1}(1)} = \frac{H_F^{-1}(v)}{\text{mean value}}
\]

...(3)

In the case of Rayleigh distribution, the TTT transformation can be obtained as

\[
F(t) = 1 - e^{-kt^2/2} \quad ; \quad 0 \leq t \leq \infty
\]

...(4)

The inverse function of \( F \) is give by

\[
F^{-1}(v) = \frac{1}{k} \left[ -\log(1 - v) \right]^{1/2} \quad ; \quad 0 \leq v \leq 1
\]

...(5)

Hence,

\[
H_F^{-1}(v) = \int_0^{F^{-1}(v)} \left[ 1 - F(u) \right] du
\]

\[
= \int_0^{\frac{1}{k} \left[ -\log(1-v) \right]^{1/2}} e^{-ku^2/2} du
\]

...(6)
By substituting \( \frac{ku^2}{2} = z \), we obtained

\[
H_F^{-1}(\nu) = \frac{1}{\sqrt{2k}} \int_0^\infty \frac{1}{z^2} e^{-z} \, dz
\]

Which shows that the TTT transformation of the Rayleigh distribution may be expressed by incomplete gamma function? However, several approximation formulas are available for incomplete gamma function. The mean time to system failure (MTSF) can also be obtained from \( H_F^{-1}(\nu) \) by substituting \( \nu = 1 \), i.e.

\[
H_F^{-1}(1) = \frac{1}{\sqrt{2k}} \int_0^\infty z^{1/2 - 1} e^{-z} \, dz = \sqrt{\frac{\pi}{2k}}.
\]

IV. CONSTRUCTION OF STRUCTURE FUNCTION

The state of \( i^{th} \) component can be described by a binary variable \( W_i \), where

\[
W_i = \begin{cases} 
1 & \text{if } i^{th} \text{ component is functioning} \\
0 & \text{if } i^{th} \text{ component is failed}
\end{cases}
\]

For \( i = 1, 2 \ldots, n \), where \( n \) is the number of components in the system. Furthermore, we assume that by knowing the states of all \( n \)-components, we are able to know whether the system is functioning or not. Similarly, the state of the system can, then, be described by a binary function

\[
\phi(\tilde{W}) = \begin{cases} 
1 & \text{if the system is functioning} \\
0 & \text{if the system is failed}
\end{cases}
\]

The function \( \phi(\tilde{W}) \) is known as the structure function of the system. Structure functions for some system configurations are as follows:

(i) \textbf{Series system configuration}

The structure function \( \phi(\tilde{W}) \) for the series configuration with \( n \) components is given by

\[
\phi(\tilde{W}) = W_1, W_2, \ldots, W_n = \prod_{i=1}^n W_i
\]

(ii) \textbf{Parallel system configuration}

The structure function \( \phi(\tilde{W}) \) for the parallel configuration with \( n \) components is given by

\[
\phi(\tilde{W}) = 1 - \prod_{i=1}^n [1 - W_i] = \prod_{i=1}^n W_i
\]

where \( \prod \) is read as "ip".
(iii) m-out of n system configuration

The structure function $\varphi(\tilde{W})$ of a k-out of n configuration is given by

$$\varphi(\tilde{W}) = \begin{cases} 
1 & \text{if } \sum_{i=1}^{n} W_i \geq m \\
0 & \text{if } \sum_{i=1}^{n} W_i < m 
\end{cases}$$

$$= (W_1 W_2 \ldots W_k) \bigvee (W_1 W_2 \ldots W_{m-1} W_{m+1}) \bigvee \ldots \bigvee (W_{n-m+1} \ldots W_n) \quad \ldots(11)$$

In particular, the structure function of 2 out of 3 configuration may also be written as

$$\varphi_{2-3}(\tilde{W}) = W_1 W_2 \bigvee W_1 W_3 \bigvee W_2 W_3$$

$$= W_1 W_2 + W_1 W_3 + W_2 W_3 - 2W_1 W_2 W_3 \quad \ldots(12)$$

Since $W_i$ is a binary variable, $W_i^j = W_i$ for all i and j. For all above standard configurations, the construction of the structure function is simple.

(iv) Non-Series Parallel Complex Configuration

The construction of the structure function for non-series parallel complex configuration is not so easier and becomes more and more cumbersome as the complexity of the configuration increases. In such case, a need of simpler technique is necessarily required. An alternative easier technique based on simple logical reasoning known as Boolean function technique is highlighted here, as used to find out the structure function for the below non-series parallel complex system configuration.

The minimal path sets for the above complex configuration (as shown in Fig. 3) are:

$$P_1 = \{1, 4, 6\}, P_2 = \{3, 5, 6\}, P_3 = \{2, 4, 6\}$$

$$P_4 = \{2, 5, 6\}$$

The logical matrix for the successful operation of the considered complex configuration can be expressed as

$$h(W_1 W_2 \ldots W_6) = \begin{pmatrix}
W_1 & W_4 & W_6 \\
W_3 & W_5 & W_6 \\
W_2 & W_4 & W_6 \\
W_2 & W_5 & W_6
\end{pmatrix} \quad \ldots(13)$$
By algebra of logic equation (13) can be written as

$$h(W_1, W_2, \ldots, W_6) = |W_6, g(W_1, W_2, \ldots, W_5)|$$

...(14)

where,

$$g(W_1, W_2, \ldots, W_5) = \begin{vmatrix} W_1 & W_4 \\ W_3 & W_5 \\ W_2 & W_4 \\ W_2 & W_5 \end{vmatrix}$$

...(15)

Substituting $M_1 = W_1 W_4$, $M_2 = W_3 W_5$, $M_3 = W_2 W_4$, $M_4 = W_2 W_5$. Hence, we get the expression

$$g(W_1, W_2, \ldots, W_5) = \begin{vmatrix} M_1 \\ M_1 M_2 \\ M_1 M_2 M_3 \\ M_1 M_2 M_3 M_4 \end{vmatrix}$$

...(16)

By algebra of logic, we have

$$M_1' = \begin{vmatrix} W_1' \\ W_1 W_4 \end{vmatrix}, \quad M_2' = \begin{vmatrix} W_3' \\ W_3 W_5 \end{vmatrix}, \quad M_3' = \begin{vmatrix} W_2' \\ W_2 W_4 \end{vmatrix}$$

where $W_i'$ is the complement of $W_i$ ($W_i' = 1 - W_i$); $i = 1, 2, \ldots, 6$

$$M_1' M_2 = \begin{vmatrix} W_1' \\ W_1 W_4 \end{vmatrix} \land \begin{vmatrix} W_3' \\ W_3 W_5 \end{vmatrix} = \begin{vmatrix} W_1' W_3 W_5 \\ W_1 W_3 W_4 W_5 \end{vmatrix}$$

$$M_1' M_2 M_3 = \begin{vmatrix} W_1' \\ W_1 W_4 \end{vmatrix} \land \begin{vmatrix} W_3' \\ W_3 W_5 \end{vmatrix} \land \begin{vmatrix} W_2' \\ W_2 W_4 \end{vmatrix} = \begin{vmatrix} W_1' W_2 W_3 W_4 W_5 \\ W_1 W_2 W_3 W_4 W_5 \end{vmatrix}$$

$$M_1' M_2' M_3 M_4 = \begin{vmatrix} W_1' \\ W_1 W_4 \end{vmatrix} \land \begin{vmatrix} W_3' \\ W_3 W_5 \end{vmatrix} \land \begin{vmatrix} W_2' \\ W_2 W_4 \end{vmatrix} = \begin{vmatrix} W_1' W_2 W_3 W_4 W_5 \\ W_1' W_2 W_3 W_4 W_5 \end{vmatrix}$$
\[
\begin{vmatrix}
W_1' & W_4 \\
W_1 & W_4'
\end{vmatrix} \land \begin{vmatrix}
W_2 & W_3' & W_4 & W_5 \\
W_1 & W_2 & W_3' & W_4 & W_5
\end{vmatrix} =
\begin{vmatrix}
W_1' & W_2 & W_3' & W_4' & W_5 \\
W_1 & W_2 & W_3' & W_4' & W_5
\end{vmatrix}
\]

Hence,

\[
g(W_1 W_2 \ldots W_5) =
\begin{vmatrix}
W_1 & W_4 \\
W_1' & W_3 \\
W_1' & W_3' & W_5 \\
W_1 & W_3 & W_4' & W_5 \\
W_1' & W_2 & W_3' & W_4' & W_5
\end{vmatrix} \quad \ldots (17)
\]

Thus, the equation (14) can be expressed as

\[
h(W_1 W_2 \ldots W_6) =
\begin{vmatrix}
W_1 & W_4 & W_6 \\
W_1' & W_3 & W_5 & W_6 \\
W_1 & W_3 & W_4 & W_5 & W_6 \\
W_1' & W_2 & W_3' & W_4' & W_5 & W_6 \\
W_1' & W_2 & W_3' & W_4' & W_5 & W_6
\end{vmatrix} \quad \ldots (18)
\]

\[
= W_1 W_4 W_6 + W_3 W_5 W_6 + W_2 W_4 W_6 + W_2 W_5 W_6 - \\
W_1 W_2 W_4 W_6 - W_2 W_3 W_5 W_6 - W_2 W_4 W_5 W_6 - \\
W_1 W_3 W_4 W_5 W_6 - W_1 W_2 W_3 W_4 W_5 W_6 \quad \ldots (19)
\]

V. EVALUATION OF RELIABILITY CHARACTERISTICS

To evaluate the reliability characteristics, we consider the following assumptions:

(i) We consider non-reparable components and systems, which are discarded the first time as they fail.
(ii) The binary state variable \( W_i (t) \) is defined as

\[
W_i (t) = \begin{cases} 
1 & \text{if ith component is functioning at time t} \\
0 & \text{if ith component is failed at time t}
\end{cases}
\]

(iii) The state variable at time t, \( W_1 (t), W_2 (t) \ldots W_n (t) \) are considered as being stochastically independent.

(iv) \( R_i (t) \) : Reliability of the \( i^{th} \) component at time t

\( R_s (t) \) : Reliability of the system at time t
Since the state variables \( W_i(t) \) \( \forall \ i = 1, 2 \ldots n \) are binary, then

\[
E[W_i(t)] = 0 \cdot P[W_i(t) = 0] + 1 \cdot P[W_i(t) = 1]
\]

\[
= P[W_i(t) = 1]
\]

\[
= P[i^{th} \text{ component is functioning at time } t]
\]

\[= R_i(t) \quad (\forall \ i = 1, 2 \ldots n)\]

Similarly, the system reliability at time \( t \) is expressed as

\[
R_S(t) = E[\varphi(W(t))] \quad \ldots(20)
\]

Under the above assumptions, \( R_s(t) \) will be the function only of \( R_i(t)'s, i = 1, 2 \ldots n \).

Now, let us determine the reliability and other characteristics of the pre-defined configurations.

(i) **Series System Configuration**

The system reliability of series systems configuration is given by

\[
\text{System reliability } R_s(t) = E\left[\varphi(W(t))\right]
\]

\[
= \prod_{i=1}^{n} R_i(t) \quad \ldots(22)
\]

For identical components, each having reliability \( R_c(t) \), then the system reliability will be expressed as

\[
R_s(t) = \left[R_c(t)\right]^n \quad \ldots(23)
\]

If the life time distribution of the each component is taken as Rayleigh with parameter \( k \), then

(i) \[ R_s(t) = e^{-nk^2t^2/2} \]

(ii) \[ \text{MTSF}_s = \int_{0}^{\infty} R_s(t) \ dt = \frac{\pi}{\sqrt{2nk}} \]

(iii) The series system life time distribution is given by

\[
f_s(t) = \frac{d}{dt} R_s(t) \ dt = nkt e^{-nk^2t^2/2}
\]

which is the Rayleigh distribution with parameter \( nk \)

(iv) The hazard rate of the series system configuration is given by

\[ h_s(t) = nk^2 \]

(2) **Parallel System Configuration**

The system reliability of parallel system configuration is given by

\[
R_p(t) = E\left[\varphi(W(t))\right]
\]
In case of identical components each life time follows Rayleigh distribution with parameter $k$, then

\[(i) \quad R_p(t) = 1 - \left[ 1 - e^{-kt^2/2} \right]^n \]

\[(ii) \quad MTSF_p = \int_0^\infty R_p(t) \, dt = \sum_{r=1}^{n} (-1)^r \binom{n}{r} \frac{\pi}{\sqrt{2rk}} \]

\[(iii) \quad \text{The parallel system life time distribution is given by} \]
\[f_p(t) = nkt e^{-kt^2/2} \left[ 1 - e^{-kt^2/2} \right]^{n-1} \]

\[(iv) \quad \text{The hazard rate of the parallel system is given by} \]
\[h_p(t) = \frac{nkt e^{-kt^2/2} \left[ 1 - e^{-kt^2/2} \right]^{n-1}}{1 - \left[ 1 - e^{-kt^2/2} \right]^n} \]

\[\text{(3) m-out of n-system configuration} \]

\[\varphi(W(t)) = \begin{cases} 1 & \text{if } \sum W_i(t) \geq m \\ 0 & \text{if } \sum W_i(t) < m \end{cases} \]

Let 
\[W_i(t) = y(t), \text{ then} \]

Hence,
\[P[Y(t) = y] = \binom{n}{y} \left[ R_c(t) \right]^y \left[ 1 - R_c(t) \right]^{n-y} \]

Thus, the reliability of a m-out of n system configuration with identical components is given by
\[R_{mn}(t) = P[Y(t) \geq m] = \sum_{y=m}^{n} \binom{n}{y} \left[ R_c(t) \right]^y \left[ 1 - R_c(t) \right]^{n-y} \]

When each components in the system follows Rayleigh life time distribution with parameter $k$, then

\[(i) \quad R_{mn}(t) = \sum_{y=m}^{n} \binom{n}{y} \left[ e^{-kt^2/2} \right]^y \left[ 1 - e^{-kt^2/2} \right]^{n-y} \]

\[= \sum_{y=m}^{n} \sum_{j=0}^{n-y} (-1)^j \binom{n-y}{j} \binom{n}{y} \left( \frac{kt^2}{2} \right)^{(y+j)} \]
(ii) \( \text{MTSF}_{mn} = \int_{0}^{\infty} R_{mn}(t) \, dt \)

\[
= \sum_{y=m}^{n} \left( \sum_{j=0}^{n-y} (-1)^{j} \begin{pmatrix} n-y \cr y \end{pmatrix} \begin{pmatrix} n-y \cr j \end{pmatrix} \right) \int_{0}^{\infty} e^{-\frac{kt^2}{2}(y+j)} \, dt
\]

\[
= \sum_{y=m}^{n} \left( \sum_{j=0}^{n-y} (-1)^{j} \begin{pmatrix} n-y \cr y \end{pmatrix} \begin{pmatrix} n-y \cr j \end{pmatrix} \right) e^{-\frac{kt^2(y+j)}{2}} \int_{0}^{\infty} e^{-\frac{kz(y+j)}{2}} \, dz
\]

\[
= \sum_{y=m}^{n} \left( \sum_{j=0}^{n-y} (-1)^{j} \begin{pmatrix} n-y \cr y \end{pmatrix} \begin{pmatrix} n-y \cr j \end{pmatrix} \right) \frac{\Gamma(1/2)}{\sqrt{2k(y+j)}} \sqrt{\pi} \frac{\Gamma(1/2)}{\sqrt{2k(y+j)}}
\]

The m-out of n system's life distribution will be

(iii) \( f_{mn}(t) = -\frac{d}{dt} R_{mn}(t) \)

\[
= \sum_{y=m}^{n} \left( \sum_{j=0}^{n-y} (-1)^{j} \begin{pmatrix} n-y \cr y \end{pmatrix} \begin{pmatrix} n-y \cr j \end{pmatrix} \right) \left[ -\frac{d}{dt} e^{-\frac{kt^2(y+j)}{2}} \right]
\]

\[
= \sum_{y=m}^{n} \left( \sum_{j=0}^{n-y} (-1)^{j} \begin{pmatrix} n-y \cr y \end{pmatrix} \begin{pmatrix} n-y \cr j \end{pmatrix} \right) \left\{ -\frac{kt^2(y+j)}{2} e^{-\frac{kt^2(y+j)}{2}} \right\}
\]

Thus, the hazard rate of the m-out of n-system's life time distribution will be

(iv) \( h_{mn}(t) = \frac{\sum_{y=m}^{n} \left( \sum_{j=0}^{n-y} (-1)^{j} \begin{pmatrix} n-y \cr y \end{pmatrix} \begin{pmatrix} n-y \cr j \end{pmatrix} \right) \left\{ -\frac{kt^2(y+j)}{2} e^{-\frac{kt^2(y+j)}{2}} \right\}}{\sum_{y=m}^{n} \left( \sum_{j=0}^{n-y} (-1)^{j} \begin{pmatrix} n-y \cr y \end{pmatrix} \begin{pmatrix} n-y \cr j \end{pmatrix} \right) e^{-\frac{kt^2(y+j)}{2}}}
\]

(4) Non-series parallel complex system configuration

Here \( R_{nsp}(t) = \mathbb{E} \left[ h(W(t)) \right] \)

\[
= \mathbb{E}[W_1(t)W_4(t)W_6(t) + W_3(t)W_5(t)W_6(t) + W_2(t)W_4(t)W_6(t)]
\]
In case of identical components, each having the life time distribution as Rayleigh with parameters $k$, then

(i) \[ R_{\text{ns}}(t) = 4e^{-at^2} - 3e^{-bt^2} - e^{-ct^2} + e^{-dt^2} \] ... (27)

where \( a = 3k/2, \ b = 4k/2, \ c = 5k/2, \ d = 6k/2 \)

(ii) \[
MTSF_{\text{ns}} = 4\sqrt{6k} - 3\sqrt{8k} - \sqrt{10k} + \sqrt{12k}
\]

\[
= \int_0^\infty R_{\text{ns}}(t) \, dt
\]

\[
= \int_0^\infty \left[ 4e^{-\frac{3kt^2}{2}} - 3e^{-\frac{4kt^2}{2}} - e^{-\frac{5kt^2}{2}} + e^{-\frac{6kt^2}{2}} \right] dt
\]

\[
= \frac{1}{2k\sqrt{2k}} \left[ \int_0^\infty \left( 4e^{-3u} - 3e^{-4u} - e^{-5u} + 6e^{-6u} \right) \frac{1}{u^2} - 1 \, du \right]
\]

(iii) The non-series parallel system's life time distribution will be

\[
f_{\text{ns}}(t) = -\frac{d}{dt} R_{\text{ns}}(t)
\]

\[
= kt \left[ 12e^{-at^2} - 12e^{-bt^2} - 5e^{-ct^2} + 6e^{-dt^2} \right]
\]

(iv) The hazard rate of the non-series parallel complex system configuration will be
VI. MAXIMUM LIKELIHOOD ESTIMATION

Let $x_{(1)}, x_{(2)}, ..., x_{(r)}$ ($r \leq n$) be a type II censored sample from Rayleigh distribution given in (1). Let us make a transformation as

$$y_{(i)} = \frac{x_{(i)}^2}{2}$$

...(28)

Here, the likelihood function of the sample in terms of $y_{(i)}$'s is given by

$$L = \frac{n!}{n-r!} k^r e^{-k \left[ \sum_{i=1}^{r} y_{(i)} + (n-r)y_{(r)} \right]}$$

...(29)

$$0 < y_{(1)} < y_{(2)} < ... < y_{(r)} < \infty$$

On taking log of both sides, we get

$$\log L = \log C + r \log k - k S_r$$

...(30)

where

$$C = \frac{n!}{n-r!}$$

and

$$S_r = \sum_{i=1}^{r} y_{(i)} + (n-r)y_{(r)}$$

differentiate both sides of (30) and putting $\frac{\partial \log L}{\partial k} = 0$,

we get the expression

$$\frac{r}{k} - S_r = 0$$

$$\Rightarrow \hat{k} = \frac{r}{S_r}$$

...(31)

For finding the distribution of $\hat{k}$, we have the following theorem:

6.1 **Theorem:**

Define $Z_{(i)} = (n - i + 1) (x_{(i)}^2 - x_{(i-1)}^2)$; $\forall i = 1, 2, ..., r$

$$= (n - i + 1) (y_{(i)} - y_{(i-1)})$; $\forall i = 1, 2, ..., r$ and $y_{(0)} = 0$

Then, (i) $\sum_{i=1}^{r} Z_{(i)} = \sum_{i=1}^{r} y_{(i)} + (n-r)y_{(r)} = S_r$

(ii) $Z_{(i)}$'s are i.i.d. exponential with parameter $k$

(iii) $\sum_{i=1}^{r} Z_{(i)} = S_r \sim G(k, r)$
Proof:

(i) \[ \sum_{i=1}^{r} Z_i = ny_{(1)} - ny_{(0)} + (n-1)y_{(2)} - (n-1)y_{(1)} + (n-2)y_{(3)} - (n-2)y_{(2)} - \ldots - y_{(r-1)} + (n-r)y_{(r)} = y_{(1)} + y_{(2)} + \ldots + y_{(r-1)} + (n-r+1)y_{(r)} \] 

\[ = \sum_{i=1}^{r} y_{(i)} + (n-r)y_{(r)} \quad \ldots \quad (32) \]

(ii) The joint p.d.f. of \( y_{(1)}, y_{(2)}, \ldots, y_{(r)} \) is given by

\[ f(y_{(1)}, y_{(2)}, \ldots, y_{(r)}) = \frac{n!}{n-r!} k^r e^{-k \left( \sum_{i=1}^{r} y_{(i)} + (n-r)y_{(r)} \right)} \]

The Jacobian of the transformation is given by

\[ \frac{1}{J} = \frac{\partial (Z_1, Z_2, \ldots, Z_r)}{\partial (y_{(1)}, y_{(2)}, \ldots, y_{(r)})} \]

\[ = \begin{vmatrix} n & 0 & 0 & \ldots & 0 \\ -(n-1) & (n-1) & 0 & \ldots & 0 \\ 0 & -(n-2) & (n-2) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & -(n-r+1) & (n-r+1) \end{vmatrix} \]

\[ = n(n-1) \ldots (n-r+1) = \frac{n!}{n-r!} \]

Hence, \( |J| = \frac{(n-r)!}{n!} \)

Now, the joint p.d.f. of \( Z_1, Z_2, \ldots, Z_r \) is given by

\[ g(Z_1, Z_2, \ldots, Z_r) = f(y_{(1)}, y_{(2)}, \ldots, y_{(r)}) |J| \]
\[ k^r e^{-k \sum z_i} \]

\[ = k e^{-kz_1} e^{-kz_2} \cdots e^{-kz_r} \]

Thus, \( Z_i \) are i.i.d. exponential with parameter \( k \).

(iii) \( \sum_{i=1}^{r} Z_i = S_r \) and \( Z_i \) are i.i.d. exponential with parameter \( k \). Hence,

\[ \sum_{i=1}^{r} Z_i \sim G(k, r) \]

By above theorem, it is clear that

(a) \[ E[\hat{k}] = E \left( \frac{r}{S_r} \right) = \int_{0}^{\infty} \frac{k^r}{\Gamma(r)} e^{-ks} S_r^{r-1} \, dS_r \]

\[ = r \frac{k^r}{\Gamma(r)} \int_{0}^{\infty} e^{-ks} S_r^{(r-1)-1} \, dS_r \]

\[ = r \frac{k^r}{\Gamma(r)} \frac{\Gamma(r-1)}{k^{r-1}} \]

\[ = \left( \frac{r}{r-1} \right) k \]

Thus, \( \hat{k} \) is positively biased estimator of \( k \).

(b) \[ E[\hat{k}]^2 = E \left( \frac{r^2}{S_r^2} \right) = r^2 E \left( \frac{1}{S_r^2} \right) \]

\[ = r^2 \frac{k^r}{\Gamma(r)} \int_{0}^{\infty} e^{-ks} S_r^{(r-2)-1} \, dS_r \]

\[ = \frac{r^2}{(r-1)(r-2)} k^2 \]

(c) \[ V[\hat{k}] = E[\hat{k}]^2 - (E(\hat{k}))^2 \]

\[ = \frac{r^2 k^2}{(r-1)^2(r-2)} ; \quad r > 2 \]

(d) \[ \frac{C.V.}{E(k)} = \frac{1}{\sqrt{(r-2)}} \]

which is independent \( k \).
VII. MAXIMUM LIKELIHOOD ESTIMATION OF DIFFERENT RELIABILITY CHARACTERISTICS OF COMPONENT AND VARIOUS SYSTEM'S CONFIGURATIONS WITH RAYLEIGH LIFE TIME DISTRIBUTION

By invariance property of MLE's, the MLE's of various reliability characteristics of component and the basic system configurations are given below.

(a) Series System Configuration

\[ \hat{R}_s(t) = e^{-\frac{ntr^2}{2Sr}} \]

\[ \tilde{MTSF}_s(t) = \frac{n\pi S_r}{2nr} \]

\[ \hat{f}_s(t) = \frac{nrt}{Sr} e^{-\frac{nrt^2}{2Sr}} \]

\[ \hat{h}_s(t) = \frac{ntr}{Sr} \]

(b) Parallel System Configuration

\[ \hat{R}_p(t) = 1 - \left[ 1 - e^{-\frac{nrt^2}{2Sr}} \right]^n \]

\[ \tilde{MTSF}_p = \sum_{u=1}^{m} (-1)^u \binom{n}{u} \sqrt{nS_r} \frac{nS_r}{2ru} \]

\[ \hat{f}_p(t) = \frac{nrt}{Sr} e^{-\frac{nrt^2}{2Sr}} \left[ 1 - e^{-\frac{nrt^2}{2Sr}} \right]^{n-1} \]

\[ \hat{h}_p(t) = \frac{nrt}{Sr} e^{-\frac{nrt^2}{2Sr}} \left[ 1 - e^{-\frac{nrt^2}{2Sr}} \right]^{n-1} \]

(c) m-out of n system configuration

\[ \hat{R}_{mn}(t) = \sum_{y=m}^{n} \sum_{j=0}^{n-y} (-1)^j \binom{n}{y} \binom{n-y}{j} e^{-\frac{nrt^2}{2Sr}} \]

\[ \tilde{MTSF}_{mn}(t) = \sum_{y=m}^{n} \sum_{j=0}^{n-y} (-1)^j \binom{n}{y} \binom{n-y}{j} \sqrt{\frac{nS_r}{2(y+j)r}} \]
\[ \hat{f}_{m,n}(t) = \sum_{y=m}^{n} \sum_{j=0}^{n-y} (-1)^j \binom{n}{y} \binom{n-y}{j} \left[ -\frac{rt^2}{2S_r} (y+j) \right] \]

\[ \hat{h}_{m,n} = \frac{\hat{f}_{m,n}(t)}{\hat{R}_{m,n}(t)} \]

\[ \hat{h}_{nsp}(t) = \frac{\sum_{y=m}^{n} \sum_{j=0}^{n-y} (-1)^j \binom{n}{y} \binom{n-y}{j} (-\frac{rt^2}{2S_r} (y+j))}{\sum_{y=m}^{n} \sum_{j=0}^{n-y} (-1)^j \binom{n}{y} \binom{n-y}{j} (-\frac{rt^2}{2S_r} (y+j))} \]

(d) **Non-series Parallel Complex Structure System**

\[ \hat{R}_{nsp}(t) = 4e^{-3t^2/2S_r} - 3e^{-4t^2/2S_r} - e^{-5t^2/2S_r} + e^{-6t^2/2S_r} \]

\[ \text{MTSF}_{nsp} = 4 \sqrt{\frac{\pi S_r}{6r}} - 3 \sqrt{\frac{\pi S_r}{8r}} - \sqrt{\frac{\pi S_r}{10r}} + 4 \sqrt{\frac{\pi S_r}{12r}} \]

\[ \hat{f}_{nsp}(t) = \frac{rt}{S_r} \left[ 12e^{-3t^2/2S_r} - 12e^{-4t^2/2S_r} - 5e^{-5t^2/2S_r} + 6e^{-6t^2/2S_r} \right] \]

\[ \hat{h}_{nsp}(t) = \frac{rt}{S_r} \left[ 12e^{-3t^2/2S_r} - 12e^{-4t^2/2S_r} - 5e^{-5t^2/2S_r} + 6e^{-6t^2/2S_r} \right] \]

**VIII. NUMERICAL EXPLORATIONS**

In the previous section, we have obtained the maximum likelihood estimator of the parameter 'k' of Rayleigh lifetime model by using type II censored sample. Further, by using invariant property of MLE's, the MLE's of various reliability characteristics of series, parallel, m out of n and non-series parallel complex system have also been derived in previous section.

To study the behaviour of the MLE's of reliability functions, MTSF and hazard rate functions, we have generated a sample of size 20 from Rayleigh distribution with k = 0.5 which is arranged as follows:

<table>
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<tr>
<th>Sample Values</th>
<th>0.3761</th>
<th>0.7500</th>
<th>1.1839</th>
<th>1.3592</th>
<th>0.7961</th>
</tr>
</thead>
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<tr>
<td></td>
<td>0.5903</td>
<td>0.7705</td>
<td>1.2011</td>
<td>1.5700</td>
<td>1.8487</td>
</tr>
<tr>
<td></td>
<td>0.6298</td>
<td>1.0509</td>
<td>1.3162</td>
<td>1.5319</td>
<td>1.8802</td>
</tr>
<tr>
<td></td>
<td>0.6461</td>
<td>1.1162</td>
<td>1.3545</td>
<td>1.7172</td>
<td>1.8889</td>
</tr>
</tbody>
</table>

The actual and MLE's values of reliability functions, MTSF and hazard rate functions of component, series, parallel, 2 out of 3 and non series parallel complex system have been obtained through a C++ computer programme. The results are given in tables 1-3. From tables it is clear that the MLE of k has very little bias as it is based on single sample.
Also, the MLE's of different reliability characteristics have little bias. It is also observed as time increases the actual and MLE's values of reliability functions as well as MTSF increase while the value of hazard rates decreases. The overall study of the observations given in Tables 1-3 shows that MLE's are very close to the actual values. Hence, in the present study the use of MLE's is strongly suggested.

REFERENCES

Table 1: \((u = 20, r = 10, t = 1, n = 3, m = 4)\)

| Reliability characteristics | Actual | MLE   | |Bias|
|-----------------------------|--------|-------|--------|
| K                           | 0.50000 | 0.50510 | 0.0051 |
| \(R_c(t)\)                  | 0.778801 | 0.776817 | 0.001934 |
| MTSF\(_c\)                  | 1.77245 | 1.76348 | 0.00897 |
| \(h_c(t)\)                  | 0.50000 | 0.50510 | 0.0051 |
| \(R_s(t)\)                  | 0.472367 | 0.468767 | 0.0036 |
| MTSF\(_s\)                  | 1.02333 | 1.018150 | 0.00518 |
| \(h_s(t)\)                  | 1.50000 | 1.515300 | 0.0153 |
| \(R_p(t)\)                  | 0.989177 | 0.98883 | 0.000294 |
| MTSF\(_p\)                  | 2.58075 | 2.56768 | 0.01307 |
| \(h_p(t)\)                  | 0.0577844 | 0.0592916 | 0.0015072 |
| \(R_{mn}(t)\)               | 0.874859 | 0.872802 | 0.002057 |
| MTSF\(_{mn}\)               | 1.71329 | 1.70462 | 0.00867 |
| \(h_{mn}(t)\)               | 0.460065 | 0.467639 | 0.007574 |
| \(R_{nps}(t)\)              | 0.722453 | 0.719496 | 0.002957 |
| MTSF\(_{nps}\)              | 1.36556 | 1.35865 | 0.00691 |
| \(h_{nps}(t)\)              | 0.8028979 | 0.8140114 | 0.0111135 |
### Table 2: \((u = 20, r = 10, t = 2, n = 3, m = 2)\)

| Reliability characteristics | Actual     | MLE        | |Bias| |
|-----------------------------|------------|------------|---|---|
| K                           | 0.50000    | 0.50510    | 0.0051 |
| \(R_c(t)\)                 | 0.367879   | 0.364146   | 0.003733 |
| \(MTSF_c\)                 | 1.77245    | 1.76348    | 0.00897 |
| \(h_c(t)\)                 | 1.00000    | 1.010200   | 0.0102 |
| \(R_s(t)\)                 | 0.049787   | 0.048287   | 0.0015 |
| \(MTSF_s\)                 | 1.02333    | 1.018150   | 0.00518 |
| \(h_s(t)\)                 | 3.00000    | 3.03060    | 0.010200 |
| \(R_p(t)\)                 | 0.74742    | 0.742918   | 0.004502 |
| \(MTSF_p\)                 | 2.58075    | 2.56768    | 0.01307 |
| \(h_p(t)\)                 | 0.590014   | 0.600590   | 0.010576 |
| \(R_{mn}(t)\)              | 0.306432   | 0.301234   | 0.005198 |
| \(MTSF_{mn}\)              | 1.71329    | 1.70462    | 0.00867 |
| \(h_{mn}(t)\)              | 1.675050   | 1.696538   | 0.021488 |
| \(R_{mps}(t)\)             | 0.139942   | 0.136325   | 0.003617 |
| \(MTSF_{mps}\)             | 1.365560   | 1.35865    | 0.00691 |
| \(h_{mps}(t)\)             | 2.564198   | 2.596647   | 0.032449 |
| Reliability characteristics | Actual | MLE  | |Bias|
|-----------------------------|--------|------|------|
| K                           | 0.50000 | 0.50510 | 0.0051 |
| Rₖ(t)                       | 0.105399 | 0.103008 | 0.002391 |
| MTSFₖ                       | 1.77245 | 1.76348 | 0.00897 |
| hₖ(t)                       | 1.50000 | 1.5153 | 0.0153 |
| Rₜ(t)                       | 0.0011709 | 0.001093 | 0.0000779 |
| MTSFₜ                       | 1.02333 | 1.018150 | 0.00518 |
| hₜ(t)                       | 4.50000 | 4.545900 | 0.0459 |
| Rₚ(t)                       | 0.284042 | 0.278285 | 0.005757 |
| MTSFₚ                       | 2.58075 | 2.56768 | 0.01307 |
| hₚ(t)                       | 1.33637 | 1.35387 | 0.0175 |
| Rₘₙ(t)                      | 0.0309852 | 0.0296459 | 0.0013393 |
| MTSFₘₙ                       | 1.71329 | 1.70462 | 0.00867 |
| hₘₙ(t)                      | 2.886639 | 2.918869 | 0.03223 |
| Rₙₛ(t)                      | 0.00430165 | 0.00402375 | 0.002779 |
| MTSFₙₛ                       | 1.365560 | 1.358650 | 0.00691 |
| hₙₛ(t)                      | 4.3632 | 4.411307 | 0.48107 |