Mathematical Modelling and Analysis of Blood Flow through Diseased Blood Vessels

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ABSTRACT

A mathematical model is presented to steady the Casson’s fluid through an inclined tube of non-uniform cross-section with multiple stenoses. The effects of various parameters on flow variables have been studied. The graphical representations have been made to validate the analytical findings with a view of its applicability to stenotic diseases. It is found that the flow of resistance increases with the height of the stenosis but decreases with the angle of inclination. It is also observed that the resistance to flow increases as viscosity of blood increases. The present study may be helpful for better understanding the flow characteristics of blood having multiple stenoses.

Keywords----- Blood flow, Yield stress, Axially symmetric stenoses, Cassons fluid model, Inclined tube, Blood vessels.

I. INTRODUCTION

In medicine, one of the major health hazards is atherosclerosis, which is the leading cause of death in many countries. Atherosclerosis or stenosis is a cardiovascular disease, which refers to the narrowing of arterial lumen, i.e. the inner open space or cavity of an artery due to deposition of fatty substances. Stenosis leads to an increase in the resistance to the flow and associated reduction in blood supply in the downstream which causes hypertension, myocardial infarction and cerebral strokes etc. (Biswas and Ali [3], Kumar and Diwakar [6]). Hence it is essential to study the blood flow through a stenosed artery to prevent the arterial diseases. In view of this, several theoretical and mathematical models (Agarwal and Varshney [1], Ellahi Rahman et al. [5]) have been developed to study the blood flow characteristics due to the presence of stenosis in the lumen of the blood vessel. Blood shows Newtonian fluid’s character when it flows through large diameter arteries at high shear rates, but it exhibits a non-Newtonian behavior when it flows through small diameter arteries at low shear rates. Herschel-Bulkely fluid is a non-Newtonian fluid, and in small diameter tubes blood behaves like Herschel-Bulkely fluid. The formation of stenosis in tubes may be of different shapes and they may be single or multiple. Venkatesan et al. [13] studied a mathematical model on pulsatile flow of Herschel-Bulkely fluid through stenosed arteries. Mohan et al. [8] studied the effect of paired stenosis through small artery. Bhatnagar et al. [2] studied the effects of an overlapping stenosis on blood flow characteristics in a narrow artery. All these investigations have considered the effects of stenosis through a tube of uniform cross-section. But, it is known that many ducts in physiological systems are not horizontal but have some inclination to the axis. Sankar et al. [10] studied the flow of Herschel-Bulkely fluid through an inclined tube of non-uniform cross section with multiple stenosis. Blood consists of suspension of cells in an aqueous solution, Mathi [7], suggested that blood behaves like a non-Newtonian fluid under certain conditions. A mathematical model of blood flow through an irregular arterial mild stenosis is developed by Rathod and Ravi [9] and they have studied that if the viscosity of fluid increases the velocity of fluid decreases in the presence of stenosis. Srivastava [12] developed a mathematical model for studying blood flow through a narrow artery with multiple stenosis and they have observed that stenosis height and axial velocity of flow very much influence the shear stress in a stenosed artery. Biswas and Barbhuiya [4] developed a mathematical model to study the effect of stenosis height and shape on resistance to flow for different values of yield stress. Some authors have analysed mathematical models by considering blood as a Herschel-Bulkley type non-Newtonian fluid. Shah [11], have studied two layered pulsatile flow of blood through an arterial tube by considering the core layer as Bingham plastic type fluid and the peripheral layer as Newtonian fluid. Vrema [14]
have investigated two-layered mathematical model by taking both the layers as Herschel-Bulkley type non-Newtonian fluid. In this paper we have presented a mathematical model to show the effect of primary and secondary stenosis assuming blood as Casson fluid model.

II. FORMULATION OF THE PROBLEM

Let us consider an axially symmetric, laminar, fully develop flow of the blood through a tube of non-uniform cross-section and with two stenoses (Fig.1).

Fig. 1. Geometry of an inclined tube with multiple stenosis

Cylindrical polar coordinate (z, r), with the pole located on the axis of the artery have been used to analyze the problem. It is assumed that the tube is inclined at an angle ‘α’ to the horizontal direction.

The momentum equation is given by

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \tau_{rz} \right) = - \frac{\partial p}{\partial z} + \frac{\sin \alpha}{r},$$

(1)

Where $\tau_{rz}$ is the shear stress for the Casson fluid,

The Casson’S constitutive equation describing the non-Newtonian behavior of blood may be written as

$$\tau_{rz}^1 = \left[ -\mu \frac{\partial u}{\partial r} \right]^\frac{1}{n} + \frac{1}{n} \tau_0,$n \tau_{rz} > \tau_0,$

(2)

$$\frac{\partial u}{\partial r} = 0,$n \tau_{rz} < \tau_0,$

(3)

and $F = \frac{\mu u_n}{\rho g R_0^2 + 1}$.

Here $u$ is the axial velocity, $p$ is the pressure, $\tau_0$ is the yield stress, $\mu$ is the fluid viscosity, $U$ is some characteristic velocity, $\rho$ is the density, $g$ is the acceleration due to gravity and $R_0$ is the radius of the tube.

When $\tau_{rz} < \tau_0$ i.e. the shear stress is less than the yield stress, there is a core region which flows as a plug (Fig. 1), and Eq. (3) corresponds to vanishing velocity gradient in that region. However, the fluid behavior is indicated whenever $\tau_{rz} > \tau_0$.

The boundary conditions are:
\( \tau \) is finite at \( r=0 \),
\( u=0 \) at \( r=h(z) \).

For the analysis presented in the sequel, we use the following non-dimensional variable
\[ \bar{z} = \frac{Z}{L}, \bar{\delta} = \frac{\delta}{R_0}, \bar{R}(z) = \frac{R(z)}{R_0}, \bar{p} = \frac{p}{\left( \mu U L R_0^2 \right)}, \bar{\tau}_0 = \frac{\tau_0}{\mu U R_0}, \bar{\tau}_x = \frac{\tau_{rx}}{\mu U R_0^2}, \bar{Q} = \frac{Q}{\pi R_0^2}, \]
\[ \bar{F} = \frac{F}{\mu U} \]

The geometry of the stenoses in non-dimensional form is given by
\[ h = R(z) = \begin{cases} 
R_0 & : 0 \leq z \leq d_1, \\
R_0 - \frac{\delta_1}{2} \left( 1 + \cos \frac{2\pi}{L_1} (z - d_1 - \frac{L_1}{2}) \right) & : d_1 \leq z \leq L_1, \\
R_0 - \frac{\delta_1}{L_2} \left( 1 + \cos \frac{2\pi}{L_2} (z - B_1) \right) & : B_1 - \frac{L_2}{2} \leq z \leq B_1, \\
R^*(z) - \frac{\delta_1}{2} \left( 1 + \cos \frac{2\pi}{L_2} (z - B_1) \right) & : B_1 \leq z \leq B_1 + \frac{L_2}{2}, \\
R^*(z) & : B_1 + \frac{L_2}{2} \leq z \leq B, 
\end{cases} \]

The following restrictions for mild stenoses are supposed to be satisfied:
\( \delta_1 \ll L_i \), where \( R_{out} = R(z) \) at \( z = B \).

Here \( L_i \) and \( \delta_i \) are the lengths and maximum heights of two stenoses.

### III. ANALYTICAL SOLUTION OF THE PROBLEM

Solving Eqs. (1) and (2) under the boundary conditions (2.4) and (2.5), we obtain the velocity as
\[ u = \frac{(P+f)}{\mu} \left[ \frac{-r^2 h^2}{4} + \frac{\tau_0}{(P+h)} (r-h) - \frac{2\pi}{3} \left( r^{3/2} - h^{3/2} \right) \left( \frac{\tau_0}{(P+f)} \right)^{1/2} \right] \]
for \( r_0 \leq r \leq h \),

where \( P = -\frac{\partial p}{\partial z} \) and \( f = \frac{\sin \alpha}{R_0} \).

Using the condition (3), we finally get the upper limit of the plug flow region (i.e the region between \( r = 0 \) and \( r = r_0 \) for which \( |\tau_{rx}| < \tau_0 \) as
\[ r_0 = \frac{2\tau_0}{(P+f)} \]
and using the condition \( \tau_{rx} = \tau_h \) at \( r = h \), we obtain
\[ \frac{r_0}{h} = \frac{\tau_0}{\tau_h} = \tau, \quad 0 < \tau < 1 \]
Taking \( r = r_0 \) in Eq. (8), we get the plug core velocity as
\[ u_p = \frac{(P+f)}{\mu} \left[ \frac{r_0^2}{12} - \frac{\tau_0}{2} + \frac{2}{3} \left( r_0^{3/2} - h^{3/2} \right) \left( \frac{\tau_0}{(P+f)} \right)^{1/2} \right] \]
for \( 0 \leq r \leq r_0 \).

The volumetric flow rate is defined by
\[ Q = 2r \left[ \int_{r_0}^{r} u_p r dr + \int_{r_0}^{h} u_r dr \right] \]
Substituting Eq. (8) and Eq. (11) in Eq. (12) and integrating, we finally get
\[ Q = \frac{h^3}{2p} (P + f) \left[ -\frac{1}{12h}(r_0^2 + 1) + \frac{\tau_0^2}{h} \left( 4 - \frac{1}{r_0^2} \right) + \frac{\tau_0}{(P+f)} \left( \frac{4}{3} (1 - r_0^3) - 2(1 - r_0^3) \right) - \frac{3}{2} \left( \frac{r_0}{(P+f)} \right) \frac{1}{h} \left( 1 - \tau_0^2 \right) \right] 
= 141 - r_0^2 \]

\( \frac{dp}{dz} = -P \)
The resistance to flow, \( \lambda \), is defined by
\[
\lambda = \frac{\Delta p}{Q}.
\]
Using Eq. (16) in Eq. (17), we get
\[
\lambda = -\frac{1}{Q} \int_{r_0}^{r_0 + 2}\left(\frac{2\tau}{3}(\tau_0 h)^2\left(\frac{1}{7}(1 - \tau^2_0) - \frac{1}{4}(1 - \tau_0^2)\right) + \left(\frac{16}{9}\tau_0 h\left(\frac{1}{7}(1 - \tau^2_0) - \frac{1}{4}(1 - \tau_0^2)\right) - 4\left(-\frac{1}{12h}(\tau_0^2 + 1) + \left(\frac{1}{7}(1 - \tau^2_0) - \frac{1}{4}(1 - \tau_0^2)\right)^2\right)\right)\right)dz
\]
(18)

\[\tau_{0.24h - \tau_{0.2431}} - \tau_{0.21 - \tau_{0.2 - 2 \mu Q h 312}/(-16h \tau 0.02 + 1 + r_0.32 h 2(4 - r 0))2 - f} \]

The pressure drop \( \Delta p \) across the stenosis between the cross-sections \( z = \pm L/2 \) can be obtained by integrating Eq. (14) as
\[
\Delta p = -f_{-L/2}^{+L/2}\left(\frac{2\tau}{3}(\tau_0 h)^2\left(\frac{1}{7}(1 - \tau^2_0) - \frac{1}{4}(1 - \tau_0^2)\right) + \left(\frac{16}{9}\tau_0 h\left(\frac{1}{7}(1 - \tau^2_0) - \frac{1}{4}(1 - \tau_0^2)\right) - 4\left(-\frac{1}{12h}(\tau_0^2 + 1) + \left(\frac{1}{7}(1 - \tau^2_0) - \frac{1}{4}(1 - \tau_0^2)\right)^2\right)\right)\right)dz
\]
(15)

\[\tau_{0.24h - \tau_{0.2431}} - \tau_{0.21 - \tau_{0.2 - 2 \mu Q h 312}/(-16h \tau 0.02 + 1 + r_0.32 h 2(4 - r 0))2 - f} \]

IV. RESULTS AND DISCUSSION

Stenosis is a serious cardiovascular disease. The irregular growth of stenosis affects the flow of blood in the arteries and which leads to serious circulatory disorders. Stenoses are formed by the accumulation of fats/ lipids on the inner wall of the arteries. Stenosis developed in the arteries can cause several diseases like blood pressure, atherosclerosis, heart attack and brain hemorrhage.

The effects of various parameters on the resistance to flow are computed numerically by taking
\[
\frac{R^*(z)}{R_0} = \exp\left[\beta B^2(z - B_1)^2\right]
\]

And \( d_1 = L_1 = L_2 = 0.2, B_1 = 0.8, B = 1, \beta_1 = 0.01 \).

It is observed that the resistance to flow increases with the height of both the primary and secondary stenosis (\( \delta_1, \delta_2 \)).
Figs. 2-3 shows the variation of resistance to flow with respect to height of secondary stenosis for different values of inclination (α) and height of primary and secondary stenosis δ₁ and δ₂. Fig. shows that the resistance to flow increases with the height of secondary stenosis. It is also notice that the resistance to flow also decreases with the decreasing value of inclination (α). Fig. 4 shows the variation of resistance to flow with respect to the height of the secondary stenosis for different values of parameter F. The resistance to flow increases for the increasing values of height of the secondary stenosis for F = 0.5. Fig. 5 depicts the results for the resistance to flow with different values of viscosity of blood. The resistance to flow increases as the viscosity of blood increases. It is also shown in this figure that the resistance to flow increases for the increasing value of the height of the secondary stenosis.
V. CONCLUSION

A mathematical model for the steady flow of Casson’s fluid through an inclined tube of non-uniform cross-section with multiple stenoses has been presented. Solutions have been obtained for primary and secondary stenoses. The resistance to flow increases as the secondary stenoses and viscosity of blood increases but the resistance to flow also decreases with the decreasing value of angle of inclination. The resistance to non-Newtonian fluid is more than the resistance to the Newtonian fluid. From the above discussion, it is clear that the ratio of maximum height of stenosis and radius of normal artery and viscosity of the non-Newtonian fluid are strong parameter in fencing the blood flow.

REFERENCES


