Parametric Analysis and Topological Studies on Differential Manifolds and Smooth Maps

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ABSTRACT

Every manifold can be built by a successive attachment of handles of increasing dimension. To such a structure there is associated a chain complex yielding the homology of the manifold. The chains are linear combinations of handles and the boundary operator is given by a matrix of intersection numbers. Of course, the same is true for a triangulation or a cellular decomposition, but the relation between the handle presentation and the homology structure of the manifold is very transparent geometrically. The minimal number of handles necessary to build an n-dimensional sphere is two: two n-discs glued along boundaries, if we succeed in proving that a homotopy sphere admits a presentation with the minimal number of handles determined by its homology and then it must admit a presentation with two handles. The homology of the manifold is given by chains groups and homomorphisms described by matrices of intersection numbers. It is well-known how this structure can be reduced through a sequence of algebraic operations to the most economical form, for instance, with all matrices diagonal, etc.

The main reason for introducing smooth structures was to enable us to define smooth functions on manifolds and smooth maps between manifolds. The study begins by defining smooth real-valued and vector-valued functions and then generalizes this to smooth maps between manifolds. We then focus our attention for a while on the special case of diffeomorphisms, which are bijective smooth maps with smooth inverses. If there is a diffeomorphism between two smooth manifolds, we say that they are diffeomorphic. The main objects of study in smooth manifold theory are properties that are invariant under diffeomorphisms.

Keywords---- manifolds, homotopy, homomorphism, smooth maps, smooth functions

I. INTRODUCTION

Milnor’s discovery of non-equivalent smooth structures on a topological sphere such structures was already classified through the newly invented method of surgery. In the same period Smale showed that every manifold can be constructed by successive attachment of handles and provided a method to obtain the most economical description. Also in the same period, Smale, Haefliger, and Hirsch developed the theory of imbeddings and immersions, vastly extending the foundational results of Whitney. The methods invented in these early years subsequently gave rise to a large amount of research. Thus the handle constructions of Smale were extended by Kirby and Siebenmann to topological manifolds and the method of surgery was expanded and applied successfully to a large variety of problems by Browder, Novikov, and Wall.

In the present study we introduce in moderate detail the notions of smooth manifold, sub-manifold and tangent space. A neighborhood of a sub-manifold of a smooth manifold can be fibered by planes, that is, it is a vector bundle, and that this bundle structure is unique. The concept of transversality due to Thom is also described in this study. This is the smooth counterpart of the notion of general position and is used similarly to extract from messy entanglements their essential geometric content.

In this study it is applied to prove that every function can be approximated by one with a very regular behaviour at singularities, a Morse function. It is also used to define intersection numbers. The present study constitutes a general background not only for differential topology but also for the study of Lie groups and Riemannian manifolds. The analytical means employed here have their roots in the implicit function theorem, the theory of ordinary differential equations, and the Brown-Sard Theorem. Some algebraic results in the form adapted for the purpose and collected in the appendix are used as well. Very little algebraic topology enters the picture at this stage.

The study is devoted to a description of various ways of gluing manifolds together: connected sum, connected sum along the boundary, attachment of handles etc. The presentation avoids the usual smoothing of corners. There is a brief discussion of the effect of these operations on homology; it prepares the ground for the more precise results of the following chapters. The last section describes a way to build some highly...
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The proof which the study illustrates here follows the original idea of Smale to manipulate handles, not the Morse functions, and adopts the following point of view. The homology of the manifold is given by chain groups (generated by handles) and homomorphisms described by matrices of intersection numbers. It is well-known how this structure can be reduced through a sequence of algebraic operations to the most economical form, for instance, with all matrices diagonal, etc. We try to find geometric operations on handles that are reflected by these algebraic operations on their algebraic counterparts: the generators of chain groups. The key to success is in the cancellation lemma, which together with Whitney’s method of eliminating unnecessary intersections, permits the actual geometric elimination of those handles whose presence is algebraically superfluous.

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II. SMOOTH VECTOR BUNDLES

There are two elements in the idea of a vector bundle: the local product structure, and the algebraic operation in the fibers. In the case of a vector bundle over a smooth manifold we have naturally the notion of a smooth vector bundle: one in which both of these are smooth.

To make this precise, we establish the notation first e.g. Let \( \pi: E \to M \) be an \( n \)-dimensional vector bundle over a smooth manifold \( M \) and let \( \{U_a, h_a\} \) be an atlas on \( M \) such that the bundle is trivial over each of the sets \( U_a \). (We are free to say sometimes that such type of charts are trivializing for \( E \)). Let \( \varphi_a \) be the composition of the canonical map \( \pi^{-1}(U_a) \to U_a \times \mathbb{R}^n \) with the projection on \( \mathbb{R}^n \). Then \( (\varphi_a, h_a) \) sends \( \pi^{-1}(U_a) \) homeomorphically onto an open subset of \( \mathbb{R}^n \times \mathbb{R}^m \); if these maps form a smooth atlas on \( E \), then we say that \( E \) is a smooth vector bundle.

To define a smooth \( r \)-metric, take the Whitney sum \( F \) of the bundle \( E \) with itself, and let \( \pi: F \to M \) be the projection of its total space onto \( M \). \( F \) is again a smooth bundle, we will consider the vectors in \( F \) to be pairs of vectors from the same fiber of \( E \).

III. TANGENT SPACE

The notion of a tangent space to a surface is quite intuitive, but the intuition depends strongly on the fact that a surface is a submanifold of \( \mathbb{R}^3 \). Nevertheless, it is possible to define the tangent space using only the smooth structure. We will do this now and show that the union of all tangent spaces forms a smooth vector bundle, the tangent bundle. The merit of proceeding this way is that the tangent bundle emerges as an invariant of the smooth structure.

A vector at a point is a direction and a magnitude. It is possible to translate this idea into the context of charts: the direction at a point would be a suitably defined equivalence class of smooth curves. But it is easier to adopt an “operational” point of view: A vector at a point associates to every function defined in the neighbourhood a number: its derivative in the direction of the vector. This operation has certain formal properties and our point of view will be to identify vectors at a point with an operation having these properties.

A tangent vector \( X \) at \( p \) is an operation which associates a number \( Xf \) to every smooth function \( f \) defined in a neighborhood of \( p \), and satisfies the following conditions:

(a) \( Xf \) and \( g \) agree in a neighborhood of \( p \), then \( Xf = Xg \);
(b) \( X(\lambda f + \mu g) = \lambda Xf + \mu Xg \) for every two numbers \( \lambda, \mu \);
(c) \( X(fg) = Xf(g(p)) + f(p)(Xg) \).

IV. VECTOR FIELDS

A vector field on a manifold \( M \) is a smooth map \( X: M \to TM \) such that \( \pi X = \text{id} \). If \( X \) is a vector field we will write \( X(p) = X_p \). Let \( \{U, h\} \) be a chart in \( M \) and \( X \) a vector field. Since \( X_p \in T_p M \) we have,

\[ X_p = \sum a_i(p) \partial_i \]
Since $X$ is smooth the functions $\alpha_i$ must be smooth. Conversely, if $X: M \to TM$ such is a section and it holds in every chart for some smooth functions $\alpha_i$, then, clearly, $X$ is a vector field. This implies that if $X$, $Y$ are vector fields and $\lambda \mu$ real numbers, then $\lambda X + \mu Y$ (defined by $(\lambda X + \mu Y)_p = \lambda X_p + \mu Y_p$) is a vector field. In the same way, one shows that if $f$ is a smooth function on $M$, then $fX$, defined by $(fX)_p = f(p)X_p$ is a vector field.

V. SUBMANIFOLDS

Intuitively, a submanifold $M$ of a manifold $N$ is a subset of $N$ which is a manifold and which locally looks like $\mathbb{R}^m$ in $\mathbb{R}^n$, at least if both $M$ and $N$ are closed.

Let $f: M \to N$ be a smooth map. We say that $f$ is an immersion if $Df$ is everywhere injective, a submersion if $Df$ is everywhere subjective. We say that $f$ is an imbedding if $f$ is an immersion and a topological imbedding, $M \subset N$ is a submanifold if the inclusion map is an imbedding.

Clearly, if $f: M \to N$ is an imbedding then $f(M)$ (with the differentiable structure induced by $f$) is a submanifold. This is not in general true iff is only a one-to-one immersion.

An example of some interest is as follows: Let $(\alpha$ be a real number and $f_{\alpha}$ the imbedding of $\mathbb{R}$ in $\mathbb{R}$ as the line $= \alpha$; let $\pi: \mathbb{R}^2 \to S^1x S^1$ be the covering map $(xy) \to (\exp(2\pi i \alpha), \exp(2\pi i y))$. Then, the composition $\pi f_{\alpha}$ is a immersion, which is one-to-one if $\alpha$ is irrational. But the image is then a dense subset of the torus and with the topology of a subset is not even a topological manifold.

VI. ISOTOPIES

We introduce now a notion of equivalence for imbeddings. Naturally enough, we want two imbeddings to be equivalent if one can be deformed to the other through imbeddings. The most convenient form of stating this precisely is as follows:

Definition: Let $f$, $g: M \to N$ be two imbeddings. An isotopy between $f$ and $g$ is a smooth map $F: M \times [0, 1] \to N$ such that

i. $F(x, 0) = f(x)$, $F(x, 1) = g(x)$

ii. $F_t = F|M \times [0, t]$ is an imbedding for $0 \leq t \leq 1$.

The assumption that $F(x, t)$ is defined for all $t$ is for technical con-venience; its actual behavior for $t < 0$ and $t > 1$ is of no importance: We can always assume that $F(x, t) = f(x)$ for $t \leq 0$ and $g(x)$ for $t \geq 0$. For let $\mu(t)$ be a smooth non-decreasing function such that $\mu(t) = 0$ for $t \leq 0$ and $=1$ for $t \geq 0$. Then $F(x, \mu(t))$ is an isotopy that is constantly $f(x)$ for $t \leq 1$ and $g(x)$ for $t \geq 1$.

VII. EXPONENTIAL MAP

Consider the total space $E$ of a smooth vector bundle $\xi$ with base $\mathcal{N}$ a smooth manifold, and projection $\pi$. We identify $\mathcal{N}$ with the zero section of $E$ and note that every open neighbourhood $U$ of $\mathcal{N}$ in $E$ contains a neighborhood that is a total space of a bundle: We can shrink $E$ to a subset of $U$. We want to show that in quite general situations submanifolds possess such bundle neighborhoods. For this reason we begin with a closer examination of the special case of the zero section $\mathcal{N}$ of the bundle $E$.

We will look first at the tangent bundle $TN$ as a subbundle of $T_N E$ the restriction of $TE$ to $N$. Since $\pi|N = id$, $D\pi$ is surjective on $T_N E$ and $T_N E = Ker(D\pi) \oplus TN$. If $E\pi$ is the fiber of $E$ at $p$, then its tangent space at $0$ is contained in $Ker(Dp_\pi)$, thus, for dimensional reasons, must equal it. This justifies the name for $Ker(Dp_\pi)$: the bundle tangent to fibers.

VIII. TRANSVERSALITY

The notion of transversality is a smooth equivalent of the notion of general position. For instance, two submanifolds $M^n$ and $V^r$ of $\mathbb{N}^n$, $n \leq m + r$, are transversal if their intersection looks locally like the intersection in $\mathbb{R}^n$ of the subspace of the first $m$ coordinates with the subspace of the last $r$ coordinates. This geometric idea is properly expressed as transversality of maps and defined in terms of their differentials.

The ability of deform maps to a transversal position is one of the most powerful techniques of differential topology. A general theorem in this direction is given here in 2.1; it will be in constant use in subsequent chapters.

In the present study we apply transversality to establish foundations of Morse theory of critical points of differentiable functions. The concept of transversality derives its strength from the theorem of Thom asserting that if $f: M \to N$ and $V$ is a submanifold of $\mathcal{N}$, then $f$ can be approximated by maps transversal on $V$.

IX. CONCLUSION

The first idea we shall meet is defining property of a manifold – to be able to describe points locally by $n$ real numbers, local coordinates. Then we define analytical objects such as vector fields, differential forms which are independent of the choice of coordinates. This has a double advantage: on the one hand it enables us to discuss these objects on topologically non-trivial manifolds like spheres, and on the other it also provides the language for expressing the equations of mathematical physics in a coordinate-free form, one of the fundamental principles of relativity. The most basic example of analytical techniques on a manifold is the theory of differential forms and the exterior derivative. This generalizes the grad, div and curl of ordinary three-dimensional calculus. It provides a very natural generalization of the theorems of Green and Stokes in three dimensions and also gives rise to de Rham cohomology which is an analytical way of approaching the algebraic topology of the manifold. This has been important
in an enormous range of areas from algebraic geometry to theoretical physics.

A topological space is a non-empty set X equipped with a distinguished family of subsets, called the open sets, with the following properties:
1) the empty set and the set X are both open,
2) the intersection of any finite collection of open sets is again open,
3) the union of any collection (finite or infinite) of open sets is again open.

Manifolds are sets where one can introduce coordinates. An n-dimensional manifold is a set whose coordinates. Although the terms function and map are technically synonymous, in studying smooth manifolds it is not possible to have a single coordinate system for the whole manifold and we need to consider instead overlapping patches each with its own system of coordinates. Despite the apparent complexity of the definition, it is usually not hard to prove that a particular map is smooth. There are basically only three common ways to do so:

- Write the map in smooth local coordinates and recognize its component functions as compositions of smooth elementary functions.
- Exhibit the map as a composition of maps that are known to be smooth.
- Use some special-purpose theorem that applies to the particular case under consideration.

A curve or surface can be described as having G^n continuity, n being the increasing measure of smoothness. Consider the segments either side of a point on a curve:

- G^0: The curves touch at the join point.
- G^1: The curves also share a common tangent direction at the join point.
- G^2: The curves also share a common center of curvature at the join point.

REFERENCES