Synthesis of Redundant Planar Isotropic Manipulator using Link Length Ratios

Hareesha N G¹ and K N Umesh²

¹Assistant Professor, Department of Aeronautical Engineering, Dayananda Sagar College of Engineering, Bengaluru, Karnataka, INDIA
²Professor, Department of Mechanical Engineering, PES College of Engineering, Mandya, Karnataka, INDIA

¹Corresponding Author: hareeshang@gmail.com

ABSTRACT

This research involves synthesis of planar isotropic 3R manipulator using link length ratios. A novel method is proposed to obtain exact solutions for synthesis of the manipulator using the conditions of isotropy. Two earlier methods of synthesis are used to compare the proposed method and are shown to be more appropriate and precise enough to get eight isotropic configurations.

Keywords— Planar manipulator, Kinematic Synthesis, Isotropy, Redundant Manipulator, Condition Number

I. INTRODUCTION

Isotropic configurations are considered as the best configurations within the workspace of any manipulator [1] for the following reasons – (i) best servo accuracy can be achieved, (ii) likelihood of error is minimal and equal in all directions and (iii) velocity and force transmission is uniform at these isotropic configurations for the norm input. The optimal kinematic synthesis of robotic linkages is an important task in the field of robotics. While designing such linkages, one should consider the isotropic configurations of the linkages.

Many research works are focused on kinematic isotropy since last 30 years. Condition number of a Jacobian of manipulator [1], Manipulability ellipsoid [2], Global isotropy index [3] are the few measures available on kinematic isotropy. The present work uses condition number and manipulability ellipsoid as measure of isotropy. The condition number of a Jacobian is unity and manipulability ellipsoid becomes a circle for isotropic manipulator. The condition number “one” indicates that there is no scaling of error. At isotropic configuration, manipulability ellipse becomes circle. Circle indicates that velocity distribution at the tip of the manipulator is uniform. Kinematic isotropy has been applied to planar 2 DOF [4], 3DOF [4]–[6] and 4 DOF and above [4], [5], [7] for mechanism design. Fully isotropic parallel manipulators are also designed [8]. Kinematic isotropy has been applied for optimum design of parallel manipulators [6], [9]. Isotropy of redundant manipulator is challenging compared to non-redundant manipulators since its Jacobian is not a square matrix. Finding inverse of such rectangular matrix is tedious. Nevertheless, many researchers have synthesized redundant manipulators [10]–[13] using kinematic isotropy. Most related methods are taken here for comparison with proposed method. Manja Kircanski [4] has developed a method to find all isotropic configurations of planar 2R and 3R manipulator.

The equations are developed for maximum and minimum singular values and condition number is expressed as a function of second Joint angle, θ₂, and ratio of link lengths l₁ and l₂ (i.e., k=l₁/l₂). In the case of 3R manipulator, two link length ratios, k₁ = l₁/l₂ and k₂ = l₂/l₁ are taken for synthesis. It has been shown that, there are eight isotropic configurations for planar 3R manipulator. Given the link length ratios k₁ and k₂, isotropic configurations are determined. K.Y. Tsai [5] has developed another method to find the isotropic configurations of planar 3R manipulators. Two equations are obtained, in terms of second and third Joint displacements, using the conditions of kinematic isotropy.

Sylvester’s dialytic elimination method is employed to get 32º degree polynomial with one variable. Exhaustive search is used to find the link lengths of the manipulator using the constraints: l₁+l₂+l₃ = 10, l₁ > 0, l₂ > 0, and l₁ + l₂ + l₃ ≤ 10. Since, it is a numerical search, sets of link lengths depends on the step size used for the search. If step size varies, same number of sets of link lengths (Number of combinations of link lengths which give eight
isotropic configurations) can’t be obtained. A step size of 0.1 yields, three sets of link lengths (i.e., 3.7, 3.7, 2.6; 3.8, 3.8, 2.4; 3.9, 3.9, 2.2) as shown in Figure 1 (a). Similarly, step size of 0.05 yields six sets as shown in Figure 1(b) and step size of 0.02 gives more sets as shown in Figure 1(c). Number of sets of link lengths is dependent on step size and constraints used. As the step size reduces, time taken to search the set of link lengths increases. Hence, there is a need for an alternate and quick method for solving the problem.

II. SYNTHESIS OF ISOTROPIC MANIPULATORS

A 2R manipulator is considered for obtaining the link length ratio for isotropy. Using the properties of 2R manipulator, isotropic configurations of 3R manipulators are obtained.

A. Conditions for Isotropic Manipulators

For non-redundant manipulators such as 2R manipulator as shown in Figure 2, Jacobian ‘J’ of manipulator is a square matrix. When the manipulator is in isotropic configuration, Jacobian of the manipulator becomes orthogonal and satisfies the condition:

\[J^T J = [I]\]  \(1\)

where \([I]\) is an identity matrix.

But, for redundant manipulators, condition defined in (1) is not valid, since J is a rectangular matrix. In such case, \([J]^T[J] \neq [I]\). The condition \([J]^T[J] = [I]\), means that the columns of J are orthogonal. This can only happen if J is an m×n matrix with \(n < m\). Similarly \([J][J]^T = [I]\) means that the rows of J are orthogonal, which requires \(n > m\). Where ‘m’ is the number of rows and ‘n’ is the number of columns of the Jacobian. ‘m’ defines the task space and ‘n’ defines number of Joints in the manipulator.

The conditions for isotropy of 2R manipulators may be defined as:

For orthogonality of rows of J,

\[J_1^T J_2 = J_2^T J_1 = 0\]  \(2a\)

For equality of the magnitude of columns of J,

\[||J_1|| = ||J_2||\]  \(2b\)

where \(J_1\) and \(J_2\) are the rows of the Jacobian J.

Isotropic conditions for redundant manipulators are slightly different from non-redundant manipulators.

For orthogonality of rows of J,

\[J^T J = 0\]  \(3a\)

For equality of the magnitude of columns of J,

\[||J_1|| = ||J_2||\].  \(3b\)

Using the conditions defined in (3), system of equations can be obtained.

B. Isotropy of Planar 2R manipulator

Consider a planar 2R manipulator as shown in Figure 2.
Figure 2. Planar 2R manipulator in general configuration

The Jacobian of the manipulator is,

\[
J = \begin{bmatrix}
-l_3s_1-l_2s_2-l_3s_3 & -l_2s_2-l_3s_3 & -l_3s_3 \\
l_1c_1+l_2c_2+l_3c_3 & l_2c_2+l_3c_3 & l_3c_3
\end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}
\]

(4)

where \( l_1, l_2 \) are the link lengths and \( \theta_1, \theta_2 \) are the absolute Joint angles. \( s_1 = \sin \theta_1; \ s_2 = \sin \theta_2; \ c_1 = \cos \theta_1; \ c_2 = \cos \theta_2. \)

Using \( J_1J_2^T = 0 \):

\[
(-l_1s_1-l_2s_2) \times (-l_2s_1l_2c_1) + (l_1c_1+l_2c_2l_3c_1) \times (-l_2c_1) = 0
\]

(5)

Using \( ||J_1|| = ||J_2|| \):

\[
(-l_1s_1-l_2s_2)^2 + (l_1c_1+l_2c_2l_3c_1)^2 = (-l_2s_1)^2 + (l_2c_1)^2
\]

(6)

Solving (5) and (6) simultaneously, we get,

\[
\frac{l_1}{l_2} = \sqrt{2}
\]

(7)

It can be noticed here that taking the ratio of link lengths, i.e., \( l_1/l_2 = \sqrt{2} \), isotropic configurations of 2R manipulator can be obtained. Using this property of link length ratios, we prove that the isotropic configurations of 3R manipulators can be easily determined.

C. Isotropy of Planar 3R manipulator

Consider a planar 3R manipulator as shown in Figure 3.

Figure 3. Planar 3R manipulator in general configuration

Jacobian of the manipulator ‘J’ with respect to tip of the manipulator is given in (8).

\[
J = \begin{bmatrix}
-l_3s_1-l_2s_2-l_3s_3 & -l_2s_2-l_3s_3 & -l_3s_3 \\
l_1c_1+l_2c_2+l_3c_3 & l_2c_2+l_3c_3 & l_3c_3
\end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}
\]

(8)

where \( l_1, l_2, l_3 \) are the link lengths and \( \theta_1, \theta_2 \), and \( \theta_3 \) are corresponding absolute Joint angles. \( J_1 \) and \( J_2 \) are row vectors of \( J_1; \ s_3 = \sin \theta_3; \) and \( c_3 = \cos \theta_3. \)

Using the conditions defined in (3), it can be written as,

\[
J_1J_2^T = \begin{bmatrix}
-l_1l_2s_2-l_1l_3s_3 & -2l_2^2c_2s_2-3l_3^2c_3s_3-2l_2l_3c_2s_3-2l_2l_3c_3s_2 = 0 \\
2l_2^2c_2-3l_3^2c_3 = 2l_2^2c_2s_3+3l_3^2s_3-l_1^2 = 2l_2l_3c_2-2l_3l_3s_3
\end{bmatrix}
\]

(9)

Equations (9) and (10) are two simultaneous non-linear equations consisting of three link lengths and four trigonometric functions. These unknowns should be determined in order to get the isotropic configurations of planar 3R manipulator. Since it is not possible to solve these two equations with seven unknowns, trigonometric identities are used to eliminate two variables from (9) and (10). Using,

\[
s_2 = \frac{2x}{1+x^2} \quad \text{and} \quad c_2 = \frac{1-x^2}{1+x^2}
\]

(11)

\[
s_3 = \frac{2y}{1+y^2} \quad \text{and} \quad c_3 = \frac{1-y^2}{1+y^2}
\]

(12)

where,

\[
x = \tan(\frac{\theta_1}{2}) \quad \text{and} \quad y = \tan(\frac{\theta_2}{2})
\]

(13)

Substituting for \( s_2, s_3, c_2 \) and \( c_3 \) from (11) and (12) into (9) and (10), and simplifying, it can be written, that

\[
2l_2^2 - l_1^2 + 3l_3^2 = \frac{4l_2^2(x^2 - 1)^2}{(x^2 + 1)^2} - \frac{6l_3^2(y^2 - 1)^2}{(y^2 + 1)^2} + 2l_1l_2(x^2 - 1)
\]

\[
+ \left(\frac{2l_2l_3(x^2 - 1)}{(y^2 + 1)} + \frac{4l_2l_3y(x^2 - 1)}{(y^2 + 1)(y^2 + 1)}\right) = 0
\]

(14)
Equations (14) and (15) are two highly non-linear simultaneous equations with five unknown variables, namely, \( l_1, l_2, l_3, x \) and \( y \). It is not possible to solve equations (14) and (15), unless we eliminate some variables or use some constraints. K.Y. Tsai [5] has used exhaustive search method to find link lengths \( l_1, l_2, l_3 \) which satisfy the equations (14) and (15). But, in the proposed method, keeping the link lengths ratio equal to \( \sqrt{2} \), isotropic configurations are obtained. Using,

\[
\frac{l_1}{l_3} = \frac{l_2}{l_3} = \sqrt{2}
\]  

Equation (16) will be satisfied if \( l_1 = l_2 = \sqrt{2} \) and \( l_3 = 1 \).

Thus, using these link lengths in (14) and (15),

\[
\frac{4l_2}{3}\left(x^2 + 1\right) \left(y^2 - 1\right) + \frac{4l_2}{3}y \left(x^2 + 1\right) = 0
\]

E. Advantages and limitations

The main advantages and limitations of the method are enumerated here.

Advantages:

i) Scaling factors can be used for link lengths depending upon requirement. There is no need to search for link lengths using constraints and searching algorithm. The method is completely analytical and no numerical search is required for searching link lengths. This reduces the time of computation and ensures host of unique solutions.

ii) Since the system of equations is substituted with link length, it becomes very convenient while computing the solutions.

iii) The method is faster and more accurate as the approach is completely analytical.

Limitations:

i) Different link lengths for first and second link \( (l_1 \text{ and } l_2) \) can’t be used.

ii) Different scaling factors for the link lengths \( (l_1, l_2, l_3) \) can’t be used.

III. CONCLUSIONS

This paper presents a novel methodology for synthesis of planar isotropic 3R manipulator using link length ratios equal to \( \sqrt{2} \). Eight isotropic configurations are obtained by solving the system of non-linear equations. Two approaches of synthesizing 3R manipulators are used for comparison. Advantages and limitations of earlier and proposed methods are highlighted. It is shown that proposed method is more convenient and accurate, since the system of equations is simple compared to the earlier methods. If link lengths are multiplied by any common scalar \( (c) \), same Joint angles can be obtained. Hence, there is no limitation in terms of link lengths which give isotropic configurations.
\[-12x+2\sqrt{2}x^2y^2+4\sqrt{2}x^3y+2\sqrt{2}x^4-4\sqrt{2}y^3x+4\sqrt{2}x^3y+4\sqrt{2}x^3y^2-4\sqrt{2}x^3y^2+4x^3+4x^3y^2+8x^3y^2-12xy^2-24xy^2-6y=0\]

\[-13-2\sqrt{2}x^2y^4+4\sqrt{2}x^4y^2+16\sqrt{2}x^3y^2+16\sqrt{2}x^3y+16\sqrt{2}xy^2+16\sqrt{2}xy^3-5x^4y^2+6\sqrt{2}x^2y^2+2\sqrt{2}x^2y^2-5x^4y^2+14x^4y^2+14x^4y^2+76x^3y^2-6\sqrt{2}x^2y^2-2xy^2+14x^3=0\]

\[26279936-13107200\sqrt{2}y^{20}-575668224\sqrt{2}y^{28}\]

\[-1315962880\sqrt{2}y^{20}+1458569216\sqrt{2}y^{24}+122683392\sqrt{2}y^{22}\]

\[+2549088256\sqrt{2}y^{20}+2848980992\sqrt{2}y^{28}-2848980992\sqrt{2}y^{14}\]

\[-2549088256\sqrt{2}y^{12}-122683392\sqrt{2}y^{10}+1458569216\sqrt{2}y^{8}\]

\[+1315962880\sqrt{2}y^{6}+131169152\sqrt{2}y^{4}+175636480\sqrt{2}y^{4}+575668224\sqrt{2}y^{4}\]

\[+13107200\sqrt{2}y^{1}+164626432y^{1}-5438177280y^{1}-1318060032y^{1}+26279936y^{1}\]

\[+164626432y^{1}+175636480y^{28}-1318060032y^{26}-5438177280y^{24}\]

\[+8442085376y^{22}+2797076480y^{20}+10669260800y^{18}+18214158336y^{16}\]

\[+1069269000y^{14}+2797076480y^{12}+8442085376y^{10}=0\]

\[-65536 (200\sqrt{2} - 401) (97x^2 + 12\sqrt{2} - 71) (-y^2 + 4\sqrt{2} + 7) (-17x^2 + 4\sqrt{2} + 7) (y^2 + 1)^2 - 0\]

\[-13107200\sqrt{2}x^8 + 150732800x^6 + 24117248\sqrt{2}x^6 + 249954304x^6 - 150732800x^4 - 24117248\sqrt{2}x^2y^2 + 26279936 + 13107200\sqrt{2}x^2 + 26279936y^8 = 0\]
REFERENCES