Topological Structures of Electromagnetic Field Lines in Photons

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ABSTRACT

Maxwell’s equations are one of the most fundamental equations of physics and have widespread applications. These equations were discovered by many brilliant scientists over hundreds of years and finally completed by Maxwell. The decouple solutions of these equations were used by Maxwell to predict that light is an electromagnetic wave which were confirmed by subsequent experiments by Hertz in 1886. Further experimental and theoretical work with electromagnetic fields, over a century, had done little to illuminate the fundamental nature of light. Here, we have used the divergence conditions for electric and magnetic fields in Maxwell’s equations, to show that only possible topological structures of electric and magnetic field lines, which can exist in free space are of the form of closed loops or knots. These solutions must evolve smoothly (due to nature of field lines) in free space and are similar to the solutions discovered by Raïada in 1989\cite{1}. Raïada’s topological solutions were based on Hopf fibration and evolve with time. The time evolution of these electric and magnetic field lines was found to preserve the topological invariants \cite{2}. The preservation of the topological structure of the field lines in these solutions has previously been ascribed to the fact that the electric and magnetic helicities, a measure of the degree of linking and knotting between field lines, are conserved \cite{1}. Another work \cite{3} suggested that the smooth evolution of the field lines is due to the stricter condition that the electric and magnetic fields be everywhere orthogonal ($\vec{E} \cdot \vec{B} = 0$). But, using very simple divergence arguments, we will show that the topological invariants of the electromagnetic knots is due to Maxwell’s divergence conditions.

KEYWORDS: Knots, Linking Number, Light, Maxwell’s equation, Hopf Fibration.

I. INTRODUCTION

Maxwell summarized the essence of electromagnetism in terms of four partial differential equations for electric and magnetic fields, which in vacuum (in absence of charges and currents) can be written as equation (1). These equations can be decoupled in terms of electric or magnetic fields alone. The decoupled equations of electric (or magnetic) fields are similar to wave equations with speed c. These wave equation solutions were used by Maxwell to predict that light is an electromagnetic wave. Hence, Maxwell equations unified the three major branches of physics namely, electricity, magnetism and optics into one theory; Electromagnetism. Since then, these equations have been one of the most important equations in physics and have played role in almost every area of physics. Some of the areas of physics heavily dependent on these equations are Atomic, Molecular, Quantum, Field Theory, Nuclear Physics, Solid State Physics, Relativity and Electromagnetism.

\begin{equation}
\n\nabla \cdot \vec{E} = 0
\end{equation}

\begin{equation}
\n\nabla \cdot \vec{B} = 0
\end{equation}

\begin{equation}
\n\n\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}
\end{equation}

\begin{equation}
\n\n\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}
\end{equation}

Where $\mu_0$ and $\varepsilon_0$ are absolute permeability and absolute permittivity of free space (vacuum).

They constitute a set of coupled, first-order, partial differential equations for electric and magnetic fields. The first two equations are about the divergence of the electric and magnetic fields, while the other two are about the curl of these fields. The electric and magnetic fields in vacuum are divergence-less (or solenoidal) fields. So, for any closed surface in vacuum, using Divergence Theorem (Gauss’s Theorem),

\begin{equation}
\n\int_S \vec{E} \cdot d\vec{a} = 0
\end{equation}

\begin{equation}
\n\int_S \vec{B} \cdot d\vec{a} = 0
\end{equation}

So, according to these equations, the total flux of electric and magnetic fields, for any volume in vacuum, is zero. Hence, if we try to visualize the electric and magnetic fields (in vacuum) in terms of electric and
magnetic field lines, the number of field lines entering in any volume are equal to the number of field lines coming out.

Fundamental nature of light have been a constant source of investigation since the ancient times but the real mathematical investigation for the light nature was initiated by Maxwell’s work. Using his equations for electromagnetism, he obtained the decoupled equations for electric and magnetic fields which were similar to the wave equations with speed of light. Using this insight, Maxwell proposed that light is an electromagnetic wave. This was subsequently confirmed by the investigations of Heinrich Hertz in 1886. But the fundamental nature of light has been a unsolved mystery to date and many problems related to the nature of light have not been answered in a satisfactory way. One of the very important unsolved problem about light is the structure of electric and magnetic field lines in photon. In the next section, the simple divergence argument is used to show that the structure of electric and magnetic field lines in photon (at least in vacuum) can be that of a closed knot type only.

II. TOPOLOGICAL STRUCTURE OF LIGHT

According to equation (2), the total flux of electric and magnetic fields, for any volume in vacuum, is zero. Hence, if we try to visualize the electric and magnetic fields (in vacuum) in terms of electric and magnetic field lines in free space, without any charge (or source), will be continuous. For any photon moving in free space, these conditions are also applicable. For photon to exist in vacuum (with divergence-less electric and magnetic fields), either, electric and magnetic field lines will have to extend from $-\infty$ to $\infty$ or electric and magnetic field lines will have to form closed continuous knots. For realistic electromagnetic fields (in vacuum), it is not possible (for field lines) to extend from $-\infty$ to $\infty$. Hence, electric and magnetic field line topology can be, only and only of the closed knot type.

Further, the evolution of electric and magnetic field lines in photon will be quite smooth as breaking of any field line at any point of space will be equivalent to non zero divergence of field at that space point. Moreover, two electric (or magnetic) field lines cannot cross each other as electric (magnetic) field will be undefined at the crossing point of the field lines. Hence, the smooth evolution of electric (magnetic) field lines is dictated by the divergence conditions of electric and magnetic fields and fundamental nature of field lines.

A very similar and important conclusion (obtained from very different reasoning) for the solution of Maxwell’s equation was made by Rañada in 1989 [1]. He found a very interesting solution to Maxwell’s equations which consists of electric (or magnetic) field lines forming closed loops with any two electric (or magnetic) field lines linked to each other. These solutions were based on Hopf fibration and were based on the fact that a complex function $\phi$ can be interpreted as a map between $S^3 \rightarrow S^2$. Maps of this kind can be classified in homotopy classes, labelled by a topological invariant called the Hopf index, the same property applies to complex scalar fields also. A simple expression for the linked electric and magnetic fields due to Rañada can be written [2] for scalar fields $\eta$ and $\xi$

$$B = \frac{1}{4\pi} \frac{\eta \times \partial \eta}{(1 + |\eta|^2)^2}; \quad E = \frac{1}{4\pi} \frac{\xi \times \partial \xi}{(1 + |\xi|^2)^2};$$

(1)

$$\xi(x,y,z,t) = \frac{(Ax + ty + i(3A + t(A - 1)))}{(tx - Ay + i(3A - t(A-1) + t^2))};$$

(2)

$$\eta(x,y,z,t) = \frac{(Ax + ty + i(tz - Ay))}{(Ax + ty + i(tz - Ay))};$$

(3)

Where $A = \frac{1}{2}(x^2 + y^2 + z^2 - t^2 + 1)$, and $x$, $y$, $z$, $t$ are dimensionless multiples of length scale $a$. Since both $\nabla \eta$ and $\nabla \eta^*$ are perpendicular to the lines of constant $\eta$, the magnetic field is tangential to the line of constant $\eta$. A similar argument can be made for the electric field and $\xi$ also [2].

In 2012, Arrayas and Trueba [4] found a new range of solutions of Maxwell’s equations in vacuum in which the topology of the field lines is that of the whole torus knots set. Torus knots are obtained by a continuous curves wrapped on surface of torus. Some interesting torus knot structures of electric and magnetic fields (consisting of Hopf link, Trefoil knot and Cinquefoil knot structures) have been discovered experimentally also [5]. Trefoil knot and Cinquefoil knots are Torus knots of 3 and 5 crossings. Torus knots of 3, 5 and 7 crossings, generated by Mathematica-9, are shown below.

![Figure 1: Three torus knots of 3, 5 and 7 crossings, respectively](image)

Time evolution of these topological structures of electromagnetic field lines not only preserves their topological structure but gives them the appearance of filaments that have a continuous identity in time, and evolve by stretching and deforming. This important property of knotted electromagnetic field lines is very
important for the continuous existence of light photon in vacuum and require further investigation.

It was argued in [1], that the invariance of topological structure for different times is due to the conservation of electric and magnetic helicities $h_e$, $h_m$, which are a measure of the average linking and knotting of electric/magnetic field lines. Further, using the ‘frozen field’ condition [7] and applying it to this free-space solution, it was pointed out in [3] that smooth evolution of electric and magnetic field lines is because of the everywhere orthogonality of electric and magnetic fields ($\mathbf{E} \cdot \mathbf{B} = 0$).

Since the photon is an electromagnetic field quanta in vacuum, the topology of electric (magnetic) field lines constituting photon is always of closed knots type. This conclusion is very important and can play a vital role to understand the fundamental nature of light photon. In the topological solutions of Maxwell’s equations [2] it has been proposed that only the carefully made light samples of certain types will contain the knotted structures of electric (magnetic) fields. Some of such torus knot structures have been measured experimentally too [6]. But the divergence-less nature of electric (magnetic) field ensure that light photons always contain knotted electric (magnetic) field lines with smooth evolutions of field lines. This conclusion is applicable for every type of light photon (independent of energy and formation).

III. SUMMARY AND CONCLUSIONS

The divergence-less electric and magnetic fields in free space will form continuous field lines. Since light photons also have divergence-less electric and magnetic fields, only allowed electric and magnetic field lines in photons will be continuous, and hence will form closed knot structures. These interesting knot structures of field lines will evolve smoothly, with time as two field lines cannot cross each other. The study of these field line knots and their various properties would be very important.

REFERENCES


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