The Generalized Difference Operator of the n^{th} Kind

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ABSTRACT

In this chapter, the authors extend the theory of the generalized difference Operator Δ_L to the generalized difference operator of the n^{th}kind denoted by Δ_L Where L =$L = \{l_1, l_2, \ldots l_n\}$ of positive reals$l_1, l_2, \ldots l_n$and obtain some interesting results on the relation between the generalized polynomial factorial of the first kind, n^{th}kind and algebraic polynomials. Also formulae for the sum of the general partial sums of products of several powers of consecutive terms of an Arithmetic progression in number theory are derived.

Keywords-- Difference Operator, Difference Equations, Arithmetic Progressions

I. INTRODUCTION

The theory of difference equations is based on the operator Δdefined as

(1) $\Delta x(n) = x(n + 1) - x(n)$, $n \in W$

Where W = {-3,-2,-1, 0, 1, 2, 3, · · ·,}. Even though many authors [1, 9–11], have defined Δ as

(2) $\Delta x(n) = x(n + \ell) - x(n)$, $\ell \in N$, no significant progress took place on this line. But recently, when we took up the definition of Δ as given in (2) and developed the theory of difference equations in a different direction, we obtained some interesting results in number theory. For convenience, we labelled the operator Δ defined by (2) as Δ_ℓ, $\ell \in W$, named it as the generalized difference operator and by defining its inverse Δ_l-1 ℓ we obtained many interesting results in number theory.

The theory was then extended for real $\ell \in (0, \infty)$ and $\Delta_{-\ell}x(n) = x(n-\ell) - x(n)$ and again many useful results were obtained in number theory. By extending the study for sequences of complex numbers and ℓ to be real, some new qualitative properties like rotatory, expanding and shrinking, spiral and web like were studied for the solutions of difference equations involving Δ_ℓ.

The results obtained can be found in [3–8].With this background, in this paper, we develop theory for Δ_L, the generalized difference operator of the n^{th} kind and obtain some significant results, relations and formulae in number theory using Stirling numbers of the second kind, S_r^n. Throughout this paper, we make use of the following assumptions.

(i) ℓ, ℓ1, ℓ2, . . . , ℓn are real numbers, C is the set of all complex numbers,
(ii) cj, c0j,c1j, . . . , c(n−1)j are constants, [x] = integer part of x,
(iii)where 0! = 1, r! = 1, 2, . . . r,
(iv) W (a) = {a, a + 1, a + 2, . . . }, Wℓ(j) = {j, j + ℓ, j + 2ℓ, . . . },
(v) L = {ℓ1, ℓ2, . . . , ℓn},
(vi) 0(L) = {φ}, φ denotes the empty set,
(vii) 1(L) = {{ℓ1}, {ℓ2}, . . . , {ℓn}},
(viii) 2(L) = {{ℓ1, ℓ2}, {ℓ1, ℓ3}, . . . , {ℓ1, ℓn}, {ℓ2, ℓ3}, . . . ,{ℓ2, ℓn}, . . . , {ℓn−1, ℓn}},
(ix) (n − 1) (L) = {{ℓ1, ℓ2, . . . , ℓn−1}, {ℓ1, ℓ2, . . . , ℓn−2, ℓn},. . . , {ℓ2, ℓ3, . . . , ℓn}},
(x) n(L) = {{ℓ1, ℓ2, . . . , ℓn}},
(xi) In general, r(L) = the set of all subsets of size r of the set L and
(xii) ℘(L) =nS r=0 r(L), the power set of L.

II. PRELIMINARIES

In this section, we present some basic definitions and preliminary results which will be useful for further discussion.

Definition-1: For a function u (k), $k \in [0, \infty)$, the generalized difference operator $\Delta_l$is defined by

$\Delta_l u(k) =$
$u(k + l) - u(k)$. (1)

Definition-2: The generalized difference operator of the n^{th}–kind, denoted as Δ_L for the function $u(k)$, $k \in [0, \infty)$, is defined as

$$\Delta_L u(k) = \sum_{r=-n}^{n} (-1)^{n-(-n)} \left\{ \sum_{A \in p = \cup r(L)} u\left(k + \sum_{l \epsilon A} l\right) \right\}. \qquad (2)$$

$$where\ k_L^{(t)} = k(k-l)(k-2l)(k-3l)\dots(k-(t-1)l).$$

Note that $\Delta_L = \Delta_{l_1}\Delta_{l_2}\dots\Delta_{l_n}$.

Definition-3: The generalized polynomial factorial of the n^{th} –kind, defined as

$$k_L^{(t)} = \sum_{r=1}^{n} (-1)^{n-(-n)} \left\{ \sum_{A \in r(L) \subset p(l)} \left((k + \sum_{li \in A - \{l_i\}} l)_{l_i}^{(t)} \right) \right\}, \qquad (3)$$

Definition-4: If $l \in (0,\infty)$ and $n \in N(1)$, then the inverse operator Δ_l^{-1}
Is defined as if

$$\Delta_l\left(z(k)\right) = u(k), then\ z(k) = \Delta_l^{-1}\left(u(k)\right) + c_j, \qquad (4)$$

Where c_j is a constant for all $k \in N_l(j), j = k - \left[\frac{k}{l}\right] l$.

The inverse of the gereralized difference operator of the n^{th} − kind denoted by Δ_L^{-1} is defined as if $\Delta_L z(k) = u(k)$, then

$$z(k) = \Delta_L^{-1}u(k) + c_{(n-1)j}\frac{k_{l_{n-1}}^{(n-1)}}{(n-1)!\ l_{n-1}^{n-1}} + c_{(n-2)j}\frac{k_{l_{n-2}}^{(n-2)}}{(n-2)!\ l_{n-2}^{n-2}}$$

$$+ \dots + c_{2_j}\frac{k_{l_2}^{(2)}}{(2)!\ l_2^2} + c_{1_j}\frac{k}{l} + c_{0_j}, \qquad (5)$$

where $c_{ij}^{'}s$ are constants. In general $\Delta_L^{-m} = \Delta_L^{-1}\left(\Delta_L^{-(m-1)}\right)$.

Lemma 6. If the Stirling numbers of the first kind is given by

$$s_n^n = 1, s_r^n = 0 \text{ for } r < 1\ and\ r \geq (n+1), \text{and} s_r^{n+1} = s_{r-1}^n - ns_r^n \text{ for } r \geq 1,$$

Then

$$\sum_{r=1}^{n} (s_r^n\ l^{n-r}k^r) = k_l^{(n)}. \qquad (6)$$

Lemma 5.2.7. [14] If $s_r^{n'}s$ are the stirling numbers of the second kind, then

$$k^n = \sum_{r=1}^{n} s_r^n\ l^{n-r}k_l^{(r)}. \qquad (7)$$

III. MAIN RESULTS

In this section, we present the formula for sum of general partial sums of products of consecutive terms of higher powers of an arithmetic progression.

Theorem 1

If $n \in N(2), l \in (0,\infty)$ and $k \in (nl,\infty)$, then

$$\Delta_{l,l,l,\dots,l}^{-1}u(k) = \sum_{r_n=2}^{r_n^*}\sum_{r_{n-1}=1}^{r_{n-1}^*}\sum_{r_{n-2}=1}^{r_{n-2}^*}\dots\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*} u(k - r_n l - r_{n-1}l - \dots - r_2 l - r_1 l)$$

$$+c_{(n-1)j}\left(\frac{k_l^{(n-1)}}{(n-1)!\,l^{n-1}}\right) + c_{(n-2)j}\left(\frac{k_l^{(n-2)}}{(n-2)!\,l^{n-2}}\right) + \dots + c_{1j}\left(\frac{k_l^{(1)}}{l}\right) + c_{0j}, (8)$$

where $r_n^* = \left[\frac{k}{l}\right] l, r_{n-i}^* = r_{n-(i-1)}^*, for\ i = 1,2,\dots,n-1$ and

$c_{0j}, c_{1j}, \dots, c_{(n-1)j}^{'}s$ are constants for all $k \in N_l(j), j = k - \left[\frac{k}{l}\right] l$ and when $n = 1$,

$$\Delta_l^{-1}u(k) = \sum_{r=1}^{\left[\frac{k}{l}\right]} u(k - rl) + c_{0j}.$$

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Proof:

$$\text{Since } \Delta_l \left\{ \sum_{r=1}^{\left[\frac{k}{l}\right]} u(k-rl) \right\} = \sum_{r=1}^{\left[\frac{k}{l}\right]+1} u(k+l-rl) - \sum_{r=1}^{\left[\frac{k}{l}\right]} u(k-rl) = u(k),$$

Definition 1, we can obtain

$$\sum_{r=1}^{\left[\frac{k}{l}\right]} u(k-rl) = \Delta_l^{-1} u(k) + c_{0j}. \qquad (9)$$

Since $\Delta_{l,l}^{-1} = \Delta_l^{-1}(\Delta_l^{-1})$, *by taking* Δ_l^{-1} on bothsides of (9) and again applying (9), we get

$$\sum_{r=1}^{\left[\frac{k}{l}\right]} \sum_{r=1}^{\left[\frac{k}{l}\right]-r_2} u(k - r_2 l - r_1 l) + \frac{k}{l} c_{1j} + c_{0j} = \Delta_{l,l}^{-1} u(k).$$

proceeding in this way and using the relation $\Delta_{l,l,l,\dots,l} = \Delta_l \Delta_l \dots \Delta_l$, we get(8) and hence the proof of the theorem.

Lemma 1

If m, n are positive integers, l is a real and $m > nl$, then

$$(i) \ (k-(n-1)l)^m - (n-1)(k-(n-2)l)^m + (-1)^{n-1} k^m = \frac{1}{n} \sum_{r=1}^{m} s_r^m \, l^{m-r} k_{l,l,\dots,l}^{(r)}. \qquad (10)$$

$$(ii) \Delta_{l,l,\dots,l}^{-1} k_l^{(m)} = \frac{k_{l,l,\dots,l}^{(m+2n-1)}}{n(m+1)(m+2)\dots(m+2n-1) l^{2n-1}} + c_{(n-1)j} \left(\frac{k_l^{(n-1)}}{(n-1)!\, l^{n-1}} \right) + c_{(n-2)j} \left(\frac{k_l^{(n-2)}}{(n-2)!\, l^{n-2}} \right) + \dots + c_{2_j} \frac{k_{l_2}^{(2)}}{(2)!\, l_2^2} + c_{1j} \left(\frac{k_l^{(1)}}{l} \right) + c_{0j}. \qquad (11)$$

$$(iii) \ k_l^{(m)} = \frac{1}{n} \sum_{r=1}^{m} s_r^m \, l^{m-r} \Delta_{l,l,\dots,l}^{-1} k_{l,l,\dots,l}^{(r)} \qquad (12)$$

Proof:

The proof follows from definitions 2, 3, 5 and the stirling numbers.

Theorem 2

If m, n are positive integers, l is a real and $m > nl$, then

$$\sum_{r_n=2}^{r_n^*} \sum_{r_{n-1}=1}^{r_{n-1}^*} \sum_{r_{n-2}=1}^{r_{n-2}^*} \dots \sum_{r_2=1}^{r_2^*} \sum_{r_1=0}^{r_1^*} (k - r_n l - r_{n-1} l - \dots - r_2 l - r_1 l)_l^{(m)}$$
$$= \frac{k_{l,l,\dots,l}^{(m+2n-1)}}{n(m+1)(m+2)\dots(m+2n-1) l^{2n-1}} + c_{(n-1)j} \left(\frac{k_l^{(n-1)}}{(n-1)!\, l^{n-1}} \right) + c_{(n-2)j} \left(\frac{k_l^{(n-2)}}{(n-2)!\, l^{n-2}} \right) + \dots + c_{2_j} \frac{k_{l_2}^{(2)}}{(2)!\, l_2^2} + c_{1j} \left(\frac{k_l^{(1)}}{l} \right) + c_{0j},$$

where c_{ij}'s are obtained by solving n equations by putting $k = (m+a)l + j$ for $a = n-1, n, n+1, \dots, 2n-2$

Proof. The proof follows by)(11) and Theorem 1

The following theorem gives the formula for sum of $(n-1)$times partial sums (ie., partial sums of partial sums of ... partial sums of) for products of p_i^{th}, $(i = 1,2,\dots,n)$ powers of m consecutive terms $k^{p_1}(k-l)^{p_2} \dots (k-(m-1))^{p_m}$ of an arithmetic progression $k, k-l, k-2l, \dots, j$,

where $j = k - \left[\frac{k}{l}\right] l$.

Theorem 4

Let s_r^t be the stirling numbers of the second kind, $p_1, p_2, p_3, \dots p_m$. are positive integers and $k \in [p_m l + j, \infty)$, where $p_m = p_1 + p_2 + \dots + p_m$. Then,

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$$\sum_{r_n=2}^{r_n^*}\sum_{r_{n-1}=1}^{r_{n-1}^*}\sum_{r_{n-2}=1}^{r_{n-2}^*}\dots\sum_{r_2=1}^{r_2^*}\sum_{r_1=0}^{r_1^*}\left[\prod_{t=1}^{m}(k-r_nl-r_{n-1}l-\dots-r_2l-(t-1)l)^{p_t}\right]$$
$$\sum_{i_1=0}^{p_1}\sum_{i_2=0}^{p_2}\sum_{i_3=0}^{p_3}\dots\sum_{i_{m-1}}^{p_m}\sum_{r=1}^{p_m-\sum i_s}\binom{p_2}{i_2}\binom{p_3}{i_2}\dots\binom{p_m}{i_{m-1}}(-1)^{i_1}(-2)^{i_2}\dots$$
$$\times(-(m-1))^{i_{m-1}}s_r^{p_m-\sum i_s}(l)^{p_m-(r+n)}\frac{k_l^{(r+n)}}{\prod_{i=1}^{n}(r+i)}+c_{(n-1)j}\left(\frac{k_l^{(n-1)}}{(n-1)!\,l^{n-1}}\right)$$
$$+c_{(n-2)j}\left(\frac{k_l^{(n-2)}}{(n-2)!\,l^{n-2}}\right)+\dots+c_{2_j}\frac{k_l^{(2)}}{(2)!\,l^2}+c_{1j}\left(\frac{k_l^{(1)}}{l}\right)+c_{0j},\qquad(13)$$

where $\sum i_s = i_1+i_2+\dots+i_{m-1}$ and the constants $c_{(n-1)j}, c_{(n-2)j},\dots,c_{0j}$ are given by solving the n equations obtained by putting $k=(p_m+a)l+j$ for $a=n-1,n,n+1,\dots,2n-2$ in (13)

Proof. From the Binomial Theorem and (4), we find

$$k^{p_1}(k-l)^{p_2}(k-2l)^{p_3}\dots(k-(m-1)l)^{p_m}=\sum_{i_1=0}^{p_2}\sum_{i_2=0}^{p_3}\sum_{i_3=0}^{p_4}\dots\sum_{i_{m-1}}^{p_m}\sum_{r=1}^{p_m-\sum i_s}\binom{p_2}{i_2}\binom{p_3}{i_2}\dots\binom{p_m}{i_{m-1}}(-1)^{i_1}(-2)^{i_2}\dots$$
$$(-(m-1))^{i_{m-1}}s_r^{p_m-\sum i_s}(l)^{p_m-r}\,k_l^{(r)}$$

Now applying the inverse operator of the n^{th} kind and making the substitution $k=(p_m+a)l+j$ for $a=n-1,n,n+1,\dots,2n-2$ in (14), we obtain the required result.

The following corollary shows the formula for sum of partial sums for products of p_i^{th} powers of m consecutive terms $k^{p_1}(k-l)^{p_2}\dots(k-(m-1)l)^{p_m}$ of an arithmetic progression $k, k-l, k-2l,\dots,j, where\ j=k-\left[\frac{k}{l}\right]l.$

Corollary 5. If $p_m, \sum i_s, s_r^t$, k and are as in Theorem 4,

$$\sum_{r_2=2}^{\left[\frac{k}{l}\right]}\sum_{r_1=0}^{\left[\frac{k}{l}\right]-r_2}\left[\prod_{t=1}^{m}(k-r_nl-r_{n-1}l-\dots-r_2l-(t-1)l)^{p_t}\right]$$
$$=\sum_{i_1=0}^{p_2}\sum_{i_2=0}^{p_3}\sum_{i_3=0}^{p_4}\dots\sum_{i_{m-1}}^{p_m}\sum_{r=1}^{p_m-\sum i_s}\binom{p_2}{i_2}\binom{p_3}{i_2}\dots\binom{p_m}{i_{m-1}}(-1)^{i_1}(-2)^{i_2}\dots$$
$$(-(m-1))^{i_{m-1}}s_r^{p_m-\sum i_s}(l)^{p_m-(r+2)}\times\left(\frac{k_l^{(r+2)}}{(r+1)(r+2)}\right)+\frac{c_{ij}k}{l}+c_{0j}\quad(15)$$

where c_{0j} and c_{1j} are constants obtained by solving the two simultaneous equations obtained by substituting $k=(p_m+a)l+j$ for $a=1,2$ in (15).

Proof. The proof follows by $n=2$ in Theorem 4.

The following corollary shows the formula for sum of partial sums for products of p_i^{th} powers of m consecutive terms $k^{p_1}(k-l)^{p_2}\dots(k-(m-1)l)^{p_m}$ of an arithmetic progression $k, k-l, k-2l,\dots,j, where\ j=k-\left[\frac{k}{l}\right]l.$

Corollary 6. If p_m, $\sum i_s$, s_r^n, k and l are as in theorem 4,

$$\sum_{r_3=2}^{\left[\frac{k}{l}\right]}\sum_{r_2=1}^{\left[\frac{k}{l}\right]-r_3}\sum_{r_1=0}^{\left[\frac{k}{l}\right]-r_3-r_2}\left[\prod_{t=1}^{m}(k-r_nl-r_{n-1}l-\dots-r_2l-(t-1)l)^{p_t}\right]=$$
$$\sum_{i_1=0}^{p_2}\sum_{i_2=0}^{p_3}\sum_{i_3=0}^{p_4}\dots\sum_{i_{m-1}}^{p_m}\sum_{r=1}^{p_m-\sum i_s}\binom{p_2}{i_2}\binom{p_3}{i_2}\dots\binom{p_m}{i_{m-1}}(-1)^{i_1}(-2)^{i_2}\dots(-(m-1))^{i_{m-1}}$$
$$s_r^{p_m-\sum i_s}(l)^{p_m-(r+3)}\times\left(\frac{k_l^{(r+3)}}{(r+1)(r+2)(r+3)}\right)+\frac{c_{2_j}k_l^{(2)}}{2!\,l^2}+c_{0j},(16)$$

where c_{0j}, c_{1j} and c_{2j} are constants obtained by solving the two simultaneous equations obtained by substituting $k = (p_m + a)l + j$ for $a = 2,3,4$ in (16).
Proof. The proof follows by $n = 3$ in Theorem 4.

APPLICATIONS

In this section, we present some examples to illustrate the main results. The following example is an illustration of Corollary 5.

Examples 1

Formula for sum of partial sums of products of 1st, 2nd and3rd powers of three consecutiveterms $(k(k-l2k-2l3of\ A.P.k,k-l,$
$k-2l,\dots,j, where\ \ j = k - \left[\frac{k}{l}\right]l$ is given by

$$\sum_{t=2}^{\left[\frac{k}{l}\right]}\sum_{s=0}^{\left[\frac{k}{l}\right]-t}(k-tl-sl)\,(k-l-tl-sl)^2(k-2l-tl-sl)^3 =$$
$$\frac{1}{56l^2}[\,k_l^{(8)} - (7l+j)_l^{(8)}] + \frac{1}{6l}[\,k_l^{(7)} - (7l+j)_l^{(7)}] + \frac{1}{3}[\,k_l^{(6)} - (7l+j)_l^{(6)}]$$
$$+\frac{l}{10}[\,k_l^{(5)} - (7l+j)_l^{(5)}] + \sum_{t=2}^{7}\sum_{s=0}^{7-l}(7l+j-tl-sl)\,(7l+j-l-tl-sl)^2$$
$$(7l+j-2l-tl-sl)^3$$
$$\left(\frac{k-(7l+j)}{l}\right)\left\{\sum_{s=1}^{5}(7l+j-sl)(6l+j-sl)^2(5l+j-sl)^3 - \frac{1}{56l^2}[(8l+j)_l^{(8)} - (7l+j)_l^{(8)}] - \frac{1}{6l}[(8l+j)_l^{(7)})\right.$$
$$\left. - (7l+j)_l^{(7)}] - \frac{1}{3}[(8l+j)_l^{(6)} - (7l+j)_l^{(6)}] + \frac{l}{10}[(8l+j)_l^{(5)} - (7l+j)_l^{(5)}]\right\}$$

Solution: By taking $p_1 = 1, p_2 = 2, p_3 = 3, m = 3$ in corollary 5, we find

$$\sum_{t=2}^{\left[\frac{k}{l}\right]}\sum_{s=0}^{\left[\frac{k}{l}\right]-t}(k-tl-sl)\,(k-l-tl-sl)^2(k-2l-tl-sl)^3 = \frac{k_l^{(8)}}{56l^2} + \frac{k_l^{(7)}}{6l}$$
$$+\frac{k_l^{(6)}}{3} + \frac{l\,k_l^{(5)}}{10} + c_{1_j}\frac{k}{l} + c_{0_j}.$$

Putting $k = (7l+j) and\ k = (8l+j) in\ (17), we\ get$

$$\sum_{t=2}^{7}\sum_{s=0}^{7-t}(7l+j-tl-sl)\,(7l+j-tl-sl-l)^2\,(7l+j-2l-tl-sl)^3 =$$
$$\frac{(7l+j)_l^{(8)}}{56l^2} + \frac{(7l+j)_l^{(7)}}{6l} + \frac{(7l+j)_l^{(6)}}{3} + \frac{l(7l+j)_l^{(5)}}{10} + c_{1_j}\frac{(7j+l)}{l} + c_{0_j}. (18)$$
$$\sum_{t=2}^{8}\sum_{s=0}^{8-t}(8l+j-tl-sl)\,(8l+j-tl-sl-l)^2\,(8l+j-2l-tl-sl)^3 =$$
$$\frac{(8l+j)_l^{(8)}}{56l^2} + \frac{(8l+j)_l^{(7)}}{6l} + \frac{(8l+j)_l^{(6)}}{3} + \frac{l(8l+j)_l^{(5)}}{10} + c_{1_j}\frac{(8j+l)}{l} + c_{0_j}. (19)$$

Hence, c_{0_j} and c_{1_j} are obtained by solving (18)and (19). now the proof follows by substituting the values of c_{0_j} and c_{1_j} in (17).

In particular, taking $k = 38$ and $l = 3$, we obtain

$$[(32)(29)^2(26)^3 + (29)(26)^2(23)^3 + \cdots + (8)(5)^2(2)^3] + [(29)(26)^2(23)^3 + (26)(23)^2(20)^3 + \cdots + (8)(5)^2(2)^3] + \cdots + (8)(5)^2(2)^3$$
$$= \frac{1}{504}[(38)_3^{(8)} - (23)_3^{(8)}] + \frac{1}{18}[(38)_3^{(7)} - (23)_3^{(7)}] + \frac{1}{3}[(38)_3^{(6)} - (23)_3^{(6)}] +$$
$$\frac{3}{10}[(38)_3^{(5)} - (23)_3^{(5)}] + \sum_{t=2}^{7}\sum_{s=0}^{7-t}(23-3t-3s)\,(20-3t-3s)^2\,(17-3t-3s)^3$$

$$+(5)\left\{\sum_{s=1}^{5}[(23-3s)(20-3s)^2(17-3s)^3]-\frac{1}{504}[(26)_3^{(8)}-(23)_3^{(8)}]-\frac{1}{18}[(26)_3^{(7)}-(23)_3^{(7)}]-\frac{1}{3}[(26)_3^{(6)}-(23)_3^{(6)}]-\frac{3}{10}[(26)_3^{(5)}-(23)_3^{(5)}]\right\}$$

$$=1573644096$$

The following example illustrates corollary 6.

Example 2. Formula for sum of second partial sums of products of 2^{nd} and 3^{rd} powers of two consecutive terms$(k^2(k-l)^3)$ *of A.P.* $k, k-l, k-2l, \ldots, j$, *where* $j=k-\left[\frac{k}{l}\right]l$ *is*

$$\sum_{t=2}^{\left[\frac{k}{l}\right]}\sum_{s=1}^{\left[\frac{k}{l}\right]-t}\sum_{r=0}^{\left[\frac{k}{l}\right]-t-s}(k-tl-sl-rl)^2(k-l-tl-sl-rl)^3=$$

$$\frac{1}{336l^3}[k_l^{(8)}-(7l+j)_l^{(8)}]+\frac{1}{30l^2}[k_l^{(7)}-(7l+j)_l^{(7)}]+\frac{1}{12l}[k_l^{(6)}-(7l+j)_l^{(6)}]$$
$$+\frac{l}{30}[k_l^{(5)}-(7l+j)_l^{(5)}]+\sum_{t=2}^{7}\sum_{r=0}^{7-t}(7l+j-tl-sl-rl)^2(7l+j-2l-tl-sl)^3$$
$$\left(\frac{k-(7l+j)}{l}\right)\{\sum_{t=2}^{8}\sum_{s=1}^{8-t}(8l+j-tl-sl)^2(8l+j-l-tl-sl)^3$$
$$-\frac{1}{336l^3}[(8l+j)_l^{(8)}-(7l+j)_l^{(8)}]-\frac{1}{30l^2}[(8l+j)_l^{(7)})-(7l+j)_l^{(7)}]$$
$$-\frac{1}{12l}[[(8l+j)_l^{(6)}-(7l+j)_l^{(6)}]+\frac{l}{30}[(8l+j)_l^{(5)}-(7l+j)_l^{(5)}]\}$$
$$+\frac{1}{2!\,l^2}[k_l^{(2)}-(7l+j)_l^{(2)}]+\left(\frac{k-(7l+j)}{l}\right)[(8l+j)_l^{(2)}-(7l+j)_l^{(2)}]$$
$$\left\{\sum_{s=1}^{5}(8l+j-tl)^2(8l+j-tl-l)^3-\frac{1}{336l^3}[(9l+j)_l^{(8)}-(7l+j)_l^{(8)}]-\frac{1}{30l^2}[(9l+j)_l^{(7)})-(7l+j)_l^{(7)}]-\frac{1}{12l}[(9l+j)_l^{(6)}-(7l+j)_l^{(6)}]-\frac{1}{30}[(9l+j)_l^{(5)}-(7l+j)_l^{(5)}]-2(-\frac{1}{336l^3}[(8l+j)_l^{(8)}-(7l+j)_l^{(8)}]-\frac{1}{30l^2}[(8l+j)_l^{(7)})-(7l+j)_l^{(5)})]-\frac{1}{12l}[(8l+j)_l^{(6)}-(7l+j)_l^{(6)})]-\frac{1}{30}[(8l+j)_l^{(5)}-(7l+j)_l^{(5)})]\right\}$$

solution: By taking $p_1=2, p_2=3, n=3, m=2$ in corollary 6, we find

$$\sum_{t=2}^{\left[\frac{k}{l}\right]}\sum_{s=0}^{\left[\frac{k}{l}\right]-t}\sum_{r=0}^{\left[\frac{k}{l}\right]-t-s}(k-tl-sl-rl)^2(k-l-tl-sl-rl)^3=$$
$$\frac{k_l^{(8)}}{336l^3}+\frac{k_l^{(7)}}{30l^2}+\frac{k_l^{(6)}}{12l}+\frac{k_l^{(5)}}{30}+c_{2_j}\frac{k_l^{(2)}}{2!\,l^2}+c_{1_j}\frac{k}{l}+c_{0_j}. \quad (20)$$

putting $k=(7l+j), k=(8l+j)$ *and* $k=(9l+j)$ *in* (20), *we obtain*

$$\sum_{t=2}^{7}\sum_{s=1}^{7-t}\sum_{r=0}^{7-t-s}(7l+j-tl-sl-rl)^2(7l+j-l-tl-sl-rl)^3=\frac{(7l+j)_l^{(8)}}{336l^3}+\frac{(7l+j)_l^{(7)}}{30l^2}+\frac{(7l+j)_l^{(6)}}{12l}+\frac{(7l+j)_l^{(5)}}{30}+c_{2_j}\frac{(7l+j)_l^{(2)}}{2!\,l^2}+c_{1_j}\frac{(7l+j)}{l}+c_{0_j}., (21)$$

$$\sum_{t=2}^{8}\sum_{s=1}^{8-t}\sum_{r=0}^{8-t-s}(8l+j-tl-sl-rl)^2(8l+j-l-tl-sl-rl)^3=\frac{(8l+j)_l^{(8)}}{336l^3}+\frac{(8l+j)_l^{(7)}}{30l^2}+\frac{(8l+j)_l^{(6)}}{12l}+\frac{(8l+j)_l^{(5)}}{30}+c_{2_j}\frac{(8l+j)_l^{(2)}}{2!\,l^2}+c_{1_j}\frac{(8l+j)}{l}+c_{0_j}., (22)$$

$$and \sum_{t=2}^{9}\sum_{s=1}^{9-t}\sum_{r=0}^{9-t-s}(9l+j-tl-sl-rl)^2\,(9l+j-l-tl-sl-rl)^3=\frac{(9l+j)_l^{(8)}}{336l^3}+\frac{(9l+j)_l^{(7)}}{30l^2}+\frac{(9l+j)_l^{(6)}}{12l}+\frac{(9l+j)_l^{(5)}}{30}+c_{2_j}\frac{(9l+j)_l^{(2)}}{2!\,l^2}+c_{1_j}\frac{(9l+j)}{l}+c_{0_j}.,\ (23)$$

Solving (21), (22) and (23), we can find c_{2_j}, c_{1_j} and c_{0_j}. Now, the proof follows by substituting the values of c_{2_j}, c_{1_j} and c_{0_j} in (20).

In particular, taking $k = 70$ and $l = 4$, we get

$$\{[(58)^2(54)^3+(54)^2(50)^3+\cdots+(6)^2(2)^3]+[(54)^2(50)^3+(50)^2(46)^3+\cdots+(6)^2(2)^3]\}+\{[(54)^2(50)^3+(50)^2(46)^3+\cdots+(6)^2(2)^3]+[(50)^2(46)^3+(46)^2(42)^3+\cdots+(6)^2(2)^3]\}+\cdots+(6)^2(2)^3=\frac{1}{336(4)^3}[(70)_4^{(8)}-(30)_4^{(8)}]$$
$$+\frac{1}{30(4)^2}[(70)_4^{(7)})-(30)_4^{(7)}]+\frac{1}{48}[(70)_4^{(6)}-(30)_4^{(6)}]+\frac{1}{30}[(70)_4^{(5)}-(30)_4^{(5)}]$$
$$+\sum_{t=2}^{7}\sum_{s=1}^{7}\sum_{r=0}^{7-s}(7l+j-tl-sl-rl)^2\,(7l+j-l-tl-sl-rl)^3+(10)$$
$$\{\sum_{s=1}^{8}\sum_{r=0}^{8-t}(8l+j-tl-sl)^2\,(8l+j-l-tl-sl)^3-\frac{1}{336(4)^3}[(34)_4^{(8)}-(30)_4^{(8)}]$$
$$-\frac{1}{30(4)^2}[(34)_4^{(7)})-(30)_4^{(7)}]-\frac{1}{48}[(34)_4^{(6)}-(30)_4^{(6)}]-\frac{1}{30}[(34)_4^{(5)}-(30)_4^{(5)}]\}$$
$$+\frac{1}{2!\,l^2}[(70)_4^{(2)})-(30)_4^{(2)}+(10)[(34)_4^{(2)}-(30)_4^{(2)}]]$$
$$\{\sum_{t=2}^{5}(8l+j-tl)^2(8l+j-tl-l)^3-\frac{1}{336(4)^3}[(38)_4^{(8)}-(30)_4^{(8)}]$$
$$-\frac{1}{30(4)^2}[(38)_4^{(7)})-(30)_4^{(7)}]-\frac{1}{48}[(38)_4^{(6)}-(30)_4^{(6)}]-\frac{1}{30}[(38)_4^{5})-(30)_4^{(5)}]$$
$$-2(-\frac{1}{336(4)^3}[(34)_4^{(8)}-(30)_4^{(8)}]-\frac{1}{30(4)^2}[(34)_4^{(7)})-(30)_4^{(7)}]$$
$$-\frac{1}{48}[(34)_4^{(6)}-(30)_4^{(6)}]-\frac{1}{30}[(34)_4^{(5)}-(30)_4^{(5)}])\}$$
$$=9293238528$$

IV. CONCLUSION

We conclude that the theory, the results and the applications obtained in this dissertation are derived using the generalized difference operators, their inverses and using the Stirling numbers of the first and second kinds. Theory and applications of the solutions of the generalized equations established in this dissertation, which are not developed earlier are applicable to the area in Numerical Methods.

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