ABSTRACT
This paper is related to positive solutions for higher order boundary value problems.

Keywords— Boundary, Value, Function

I. INTRODUCTION
In this Paper, we generate the existence of positive solutions for boundary value problems

1.1 LEMMA
In this section, we present the following assumptions and lemma that are used for proving our theorems.

Let \( X = C^{n-2}[0,1] \).

cone \( K = \{ y \in X : y(t) \geq 0 \} \).

Let \( \| \cdot \| \) denote the supremum norm on \( X \), and for a constant \( c \),

let \( K_c = \{ y \in K : \| y \| < c \} \).

\( \partial K_c = \{ y \in K : \| y \| = c \} \).

Let \( N = \max_{0 \leq t \leq 1} \int_0^1 G(t,s)h(s)ds \).

We use also the following assumptions:

(A1) \( \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0 \) and \( \rho = \alpha_1 \alpha_2 + \alpha_1 \beta_2 + \alpha_2 \beta_1 > 0 \);

(A2) \( f \in C([0,1] \times [0, +\infty), R) \);

(A3) \( h(t) \in C((0,1), R^+) \), \( 0 < \int_0^1 G(t,s)h(s)ds < +\infty \).
where $R^+ = [0, +\infty)$, $G(t, s)$ is the Green’s function of the problem $-x^{(n)} = 0$ with the boundary conditions (1.2) - (1.4).

It is clear that

$$g(t, s) = \frac{\partial^{n-2} G(t, s)}{\partial t^{n-2}}, \quad \rightarrow (1.1)$$

is the Green’s function of the boundary value problems

$$-x' = 0, \quad 0 < t < 1, \quad \rightarrow (1.2)$$

$$\begin{align*}
\alpha_1 x(0) - \beta_1 x'(0) &= 0, \\
\alpha_2 x(1) + \beta_2 x'(1) &= 0,
\end{align*} \quad \rightarrow (1.3)$$

given by,

$$g(t, s) = \begin{cases} 
\frac{1}{\rho} (\alpha_1 t + \beta_1) [\alpha_2 (1 - s) + \beta_2], & t \leq s; \\
\frac{1}{\rho} (\alpha_1 s + \beta_1) [\alpha_2 (1 - t) + \beta_2], & s \leq t.
\end{cases} \quad \rightarrow (1.4)$$

**LEMMA: A**

Suppose $T: X \to X$ is completely continuous. Define the operator $\theta : TX \to K$ by

$$(\theta y)(t) = \max \{ y(t), w(t) \}, \quad \text{for } y \in TX,$$

where $w(t) \in C^{n-1}[0, 1], \quad w(t) \geq 0$ is a given function. Then

$$(\theta \circ T) : X \to K$$

is also a completely continuous operator.

**II. BVP**

We suppose that (A1), (A2), (A3) hold. The following Theorems (2.1) and (2.2) give sufficient conditions which guarantee the existence of positive solutions for BVP (1.1) – (1.4).

**THEOREM: 1.2.1**

Assume there exist constants $r > M > 0$, such that
\[ 0 < \frac{M}{\min_{0 \leq r \leq 1} f(t, Mw(t))} = a < b = \frac{r}{\max_{0 \leq r \leq 1} f(t, x)} \rightarrow (1.1) \]

Then BVP (1.1) – (1.4) has at least one positive solution \( y(t) \) satisfying

\[ 0 < Mw(t) \leq y(t), \quad 0 < t < 1 \quad \text{and} \quad \|y\| < r. \rightarrow (1.2) \]

provided that \( \lambda \in [a, b) \).

**PROOF:**

We define the auxiliary function \( F(t, x) \) as

\[ F(t, x) = \begin{cases} f(t, x), & x \geq Mw(t), \\ f(t, Mw(t)), & x < Mw(t). \end{cases} \rightarrow (1.3) \]

Let the operator \( T : K \rightarrow K \) be defined by

\[ (Tx)(t) = \int_0^1 G(t, s)h(s)F(t, x(s))ds, \quad 0 \leq t \leq 1. \rightarrow (1.4) \]

Then \( T \) is on \( K \) a completely continuous operator.

Let the operator \( \theta : X \rightarrow K \) be defined by

\[ (\theta y)(t) = \max \{y(t), 0\}. \rightarrow (1.5) \]

From lemma (A),

\( \theta o T : K \rightarrow K \) is also completely continuous.

For \( x \in \partial K_r \), set

\[ J = \{ t \in [0,1] : F(t, x(t)) \geq 0 \}. \]

Then we have,

\[ (\theta o T)x(t) = \max \{(Tx)(t), 0\} \]

\[ = \max \left\{ \frac{1}{\lambda} \int_0^1 G(t, s)h(s)F(t, x(s))ds, 0 \right\} \]
\[ \leq \lambda \int_{0}^{1} G(t,s)h(s)F(t,x(s))ds \]

\[ \leq \lambda \int_{y} G(t,s)h(s)F(t,x(s))ds \]

\[ < b \max_{0 \leq s \leq 1} F(t,x) \int_{y} G(t,s)h(s)ds \]

\[ \leq Nb \max_{0 \leq s \leq 1} f(t,x) \]

\[ \leq r. \]

Then for every \( x \in \partial K_r \),

\[ (\theta o T)x(t) \neq x, \]

it follows that

\[ \text{deg}_K \{ I - \theta o T, K_r, 0 \} = 1, \]

where \( \text{deg}_K \) stands for the degree in cone \( K \).

Then \( \theta o T \) has a fixed point \( y \in K_r \).

To finish the proof based on the definition of \( F \), it suffices to show that the fixed point \( y \in K_r \) satisfying

\[ (Ty) (t) \geq Mw(t), \quad 0 \leq t \leq 1, \quad \rightarrow (1.6) \]

since \( F = f \) in the region.

In order to show (3.2.6) is hold, we first show that

\[ (Ty)^{(n-2)}(t) \geq Mw^{(n-2)}(t), \quad 0 \leq t \leq 1. \quad \rightarrow (1.7) \]

Otherwise, let

\[ u(t) = Mw^{(n-2)}(t) - (Ty)^{(n-2)}(t), \quad 0 \leq t \leq 1, \]

then there exists \( t_0 \in [0,1] \) such that

\[ u(t_0) = \max_{0 \leq s \leq 1} \{ u(t) \} = A > 0. \quad \rightarrow (1.8) \]

If \( t_0 = 0 \), then

\[ u'(t) = (n-2)Mw^{(n-3)}(t) - (n-2)(Ty)^{(n-3)}(t) \]
\[ u'(t_0) = (n-2)Mw^{(n-3)}(t_0) - (n-2)(Ty)^{(n-3)}(t_0) \]

Therefore \( u'(0) \leq 0 \).

Since both \( Mw(t) \) and \( (Ty)(t) \) satisfy the boundary condition (1.3), we have

\[
\alpha_t u(0) - \beta u'(0) \\
= \alpha_t [Mw^{(n-2)}(0) - (Ty)^{(n-2)}(0)] - \beta_t [Mw^{(n-1)}(0) - (Ty)^{(n-1)}(0)] \\
= \alpha_t Mw^{(n-2)}(0) - \alpha_t (Ty)^{(n-2)}(0) - \beta_t Mw^{(n-1)}(0) + \beta_t (Ty)^{(n-1)}(0) \\
= \alpha_t Mw^{(n-2)}(0) - \beta_t Mw^{(n-1)}(0) - \alpha_t (Ty)^{(n-2)}(0) + \beta_t (Ty)^{(n-1)}(0) \\
= 0.
\]

Therefore \( \alpha_t u(0) - \beta u'(0) = 0 \).

If \( \beta_1 = 0 \), from \( \rho > 0 \), then \( \alpha_1 > 0 \), so \( u(0) = 0 \), which contradicts to (1.2.8).

Then

\[ \alpha_1 = 0, \beta_1 > 0 \text{ and } u'(0) = 0. \quad \rightarrow \text{(1.9)} \]

Now we claim that

\[ u(t) > 0, \ t \in [0,1]. \quad \rightarrow \text{(1.10)} \]

If the assertion is false, then there is \( t_1 \in (0,1] \) such that \( u(t) > 0, t \in [0,t_1), u(t_1) = 0. \quad \rightarrow \text{(1.2.11)} \)

So for every \( t \in (0,t_1] \), from (3.2.9), we have

\[
u'(t) = u'(0) + \int_0^t u''(s) \, ds \\
= \int_0^t [Mw^{(n)}(s) - (Ty)^{(n)}(s)] \, ds \\
= -\int_0^t h(s)[M - \lambda F(t, y(s))] \, ds \\
\geq 0.
\]
That is \( u'(t) \geq 0, \ t \in [0, t_1] \), and from (1.2.11) we have the following contradiction
\[
0 = u(t_1) \geq u(0) = L > 0.
\]
Then Equ. (2.10) is hold.

If \( t_0 = 1 \), we can obtain (2.10) in a similar way.

Finally, if \( t_0 \in (0, 1) \) then \( u'(t_0) = 0 \). We are able to show that \( u'(t) \geq 0 \) respectively in \([0, t_0]\) and in \([t_0, 1]\) in the same way as the above argument. So (3.2.10) holds for all the possible cases.

Since both \( Mw(t) \) and \( (Ty)(t) \) satisfy the boundary condition (1.2)-(1.4), that is,
\[
Mw^{(i)}(0) = (Ty)^{(i)}(0) = 0, \ i = 0, 1, ..., n - 3.
\]
From (3.2.10), we have
\[
Mw(t) - (Ty)(t) = \int_0^{t_{n+3}} \int_0^{\tau_1} ... \int_0^{\tau_{n-3}} u(s)d\tau_1 ... d\tau_{n-3}
\]
\[
> 0. \quad \rightarrow (1.12)
\]
Then we have the following contradiction
\[
0 < Mw(t_0) - (Ty)(t_0)
\]
\[
= \int_0^1 G(t_0, s)h(s)Md\tau - \lambda \int_0^1 G(t_0, s)h(s)F(s, y(s))d\tau
\]
\[
= \int_0^1 G(t_0, s)h(s)[M - \lambda F(s, y(s))]d\tau
\]
\[
\leq [M - \lambda \min_{0 \leq t \leq t_1} f(t, Mw(t))] \int_0^1 G(t_0, s)h(s)d\tau
\]
\[
\leq 0.
\]
Then \((\theta o T)y = Ty = y\) and \( y \) is a solution of boundary value problem (1.1)-(1.4).

Hence the proof of the theorem.

**THEOREM: 3**

Assume \( f(t, 0) \geq 0, \ h(t)f(t, 0) \not= 0 \) and there exists \( r > 0 \), such that
Then when \( \lambda < b \) boundary value problem (1.1)-(1.4) has at least one positive solution \( y(t) \) satisfying \( 0 < \| y \| < r \).

**PROOF:**

Let \( F(t,x) = \begin{cases} f(t,x), & x \geq 0, \\ f(t,0) - x, & x < 0. \end{cases} \) \( \to (1.14) \)

Similar to the proof of Theorem (1.2.1), we show \( \theta o T \) has a fixed point \( y \in K_r \), where \( K_r \) and \( T \) defined as in Theorem (1.2.1).

Now we claim that

\[
(Ty)^{(n-2)}(t) \geq 0, \quad 0 \leq t \leq 1. \quad \to (1.15)
\]

Otherwise, then there exists \( t_0 \in [0,1] \) such that

\[
(Ty)^{(n-2)}(t_0) = \min_{0 \leq t \leq 1} \{ (Ty)^{(n-2)}(t) \} = -B < 0.
\]

We prove that

\[
(Ty)^{(n-2)}(t) < 0, \quad 0 \leq t \leq 1. \quad \to (1.16)
\]

If \( t_0 \in (0,1) \), then

\[
(Ty)^{(n-1)}(t_0) = 0.
\]

If (3.2.16) does not hold, then there is \( t_1 \in [0,t_0) \cup (t_0,1] \) satisfying

\[
(Ty)^{(n-2)}(t_1) = 0, \quad \text{and}
\]

\[
(Ty)^{(n-2)}(t) < 0, \quad t \in (t_1,t_0) \text{ or } t \in (t_0,t_1). \quad \to (1.17)
\]

Without loss of generality we suppose \( t_1 \in [0,t_0) \). Then from boundary condition (1.2), for every \( t \in (t_1,t_0) \), we have

\[
(Ty)(t) = \int_0^t \int_0^{\tau_3} \cdots \int_0^{\tau_1} (Ty)^{(n-2)}(s)dsd\tau_1\cdots d\tau_{n-3} < 0. \quad \to (1.18)
\]
From (3.2.18), for every \( t \in (t_1, t_0) \)

\[
(Ty)^{(n-1)}(t) = (Ty)^{(n-1)}(t_0) - \int_{t_0}^{t} (Ty)^{(n)}(s)\,ds
\]

\[
= \lambda \int_{t}^{t_0} h(s) F(t, y(s))\,ds
\]

\[
\geq 0.
\]

Which implies the following contradiction

\[
0 = (Ty)^{(n-2)}(t_1) = (Ty)^{(n-2)}(t_0) + \int_{t_0}^{t_1} (Ty)^{(n-1)}(s)\,ds
\]

\[
= -B - \int_{t_1}^{t_0} (Ty)^{(n-1)}(s)\,ds
\]

\[
\leq -B
\]

\[
< 0.
\]

Then (1.2.15) is hold.

So for \( t \in [0, 1] \), we have

\[
(Ty)(t) = \int_{0}^{t_0} \int_{0}^{\tau_1} \ldots \int_{0}^{\tau_{n-3}} (Ty)^{(n-2)}(s)\,ds\,d\tau_1\ldots d\tau_{n-3}
\]

\[
\geq 0.
\]

If \( t_0 = 0 \) or \( t_0 = 1 \), with use of the boundary conditions we can show the above assertion in a similar way in Theorem (1.1).

Then

\[
y = (\theta \circ T) \, y = Ty,
\]

that is \( y(t) \) is a non negative solution of boundary value problem (1.1) - (1.4) with \( 0 \leq \|y\| \leq r \).

### III. CONCLUSION

Besides \( h(t) f(t, 0) \neq 0 \) implies \( y(t) \neq 0 \) in \([0,1]\).

Therefore \( 0 \leq \|y\| \leq r \).

Hence the proof of the theorem.
REFERENCES


