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Fair Secure Dominating Set in the Corona of Graphs

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ABSTRACT

In this paper, we extend the concept of fair secure dominating sets by characterizing the corona of two nontrivial connected graphs and give some important results.

***Keywords*—** Fair Dominating Set, Secure Dominating Set, Fair Secure Dominating Set, Corona, Cartesian Product

I. INTRODUCTION

A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the vertex-set of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the edge-set of G. The elements of $V(G)$ are called vertices and the cardinality $|V(G)|$ of $V(G)$ is the order of G. The elements of $E(G)$ are called edges and the cardinality $|E(G)|$ of $E(G)$ is the size of G. If $|V(G)| = 1$, then G is called a trivial graph. If $E(G) = \emptyset$, then G is called an empty graph. The open neighborhood of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. The elements of $N_G(v)$ are called neighbors of v. The closed neighborhood of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the open neighborhood of X in G is the set $N_G(X) = \cup_{v\in X} N_G(v)$. The closed neighborhood of X in G is the set $N_G[X] = \cup_{v\in X} N_G[v] = N_G(X) \cup X$. When no confusion arises, $N_G\,[x]$ [resp. $N_G(x)$] will be denoted by $N[x]$ [resp. $N(x)$]. For the general terminology in graph theory, readers may refer to [1].

Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [2]. A subset S of $V(G)$ is a dominating set of G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The domination number $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. Related studies on domination in graphs were found in the papers [3,4,5,6,7,8,9,10,11,12].

In 2011, Caro, Hansberg and Henning [13] introduced fair domination and k-fair domination in graphs. A dominating subset S of $V(G)$ is a fair dominating set in G if all the vertices not in S are dominated by the same number of vertices from S, that is, $|N(u) \cap S| = |N(v) \cap S|$ for every two distinct vertices u and v from $V(G) \setminus S$ and a subset S of $V(G)$ is a k-fair dominating set in G if for every vertex $v \in V(G) \setminus S$, $|N(v) \cap S| = k$. The minimum cardinality of a fair dominating set of G, denoted by $\gamma_{fd}(G)$, is called the fair domination number of G. A fair dominating set of cardinality $\gamma_{fd}(G)$ is called γ_{fd}-set. Related studies on fair domination in graphs were found in the papers [14,15,16,17,18].

Other variant of domination is the secure domination in graphs. A dominating set S of $V(G)$ is a secure dominating set of G if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The minimum cardinality of a secure dominating set of G, denoted by $\gamma_s(G)$, is called the secure domination number of G. A secure dominating set of cardinality $\gamma_s(G)$ is called a γ_s-set of G. Secure dominating set was introduced by E.J. Cockayne et.al [19]. Secure dominating sets can be applied as protection strategies by minimizing the number of guards to secure a system so as to be cost effective as possible. Some variants of secure domination in graphs were found in the papers [20,22,23,24,25,26,27,28,29].

A fair dominating set $S \subseteq V(G)$ is a fair secure dominating set if for each $u \in V(G) \setminus S$ there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The minimum cardinality of a fair secure dominating set of G denoted by $\gamma_{fsd}(G)$ is called the fair secure domination number of G A fair secure dominating set of cardinality $\gamma_{fsd}(G)$ is called γ_{fsd}-set. In this paper, we extend the study of fair secure dominating set by giving the characterization of a fair secure dominating set in the corona of two nontrivial connected graphs and give some important results.

II. RESULTS

The following known results are needed in this paper.

Remark 2.1 *[13] If $G \neq \bar{K}_n$, then $\gamma_{fd}(G) = min\{\gamma_{kfd}(G)\}$, where the minimum is taken over all integers k where $1 \leq k \leq |V(G)| - 1$.*

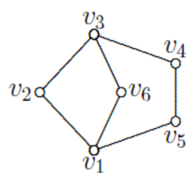

Figure 1: A graph G with $\gamma_{fsd}(G) = 3$

Example 2.2 Consider the graph in Figure 1. The sets $S_1 = \{v_1, v_5, v_6\}$, $S_2 = \{v_1, v_3, v_5\}$ and $S_3 = \{v_2, v_4, v_5, v_6\}$ are fair secure dominating sets of G. Thus S_1 or S_2 is a minimum fair secure dominating set of G. Hence, $\gamma_{fsd}(G) = 3$.

Remark 2.3 *[15] A fair secure dominating set of a graph G is a fair dominating set and a secure dominating set of G.*

Remark 2.4 *[15] Let G be any connected graph of order $n \geq 2$. Then*

(i) $1 \leq \gamma_{fsd}(G) \leq n-1$ *and*
(ii) $\gamma(G) \leq \gamma_{fd}(G) \leq \gamma_{fsd}(G)$.

Remark 2.5 *[15] Let $n \geq 2$. The $\gamma_{fsd}(K_n) = 1$.*

We need the following results for the characterization of the fair secure dominating set resulting from the corona of two graphs.

Lemma 2.6 *Let $G = x + H$ where H is a nontrivial connected graph. If S_x is a dominating set of H where $S_x \neq V(H)$, then $S = \{x\} \cup S_x$ is a secure dominating set of G.*

Proof **:** Suppose S_x is a dominating set of H. Let $S = \{x\} \cup S_x$. Then S is a dominating set of G. Since $S_x \neq V(H)$, let $u \in V(H) \setminus S_x$. Then for every $u \in V(G) \setminus S$ there exists $v \in S$ such that $uv \in E(G)$. Let $S' = (S \setminus \{v\}) \cup \{u\}$. If $v = x$, then $S' = (S \setminus \{x\}) \cup \{u\} = S_x \cup \{u\}$. Since S_x is a dominating set of H, it is a dominating set of G. Thus, S' is a dominating set of G. If $v \neq x$, then $v \in S_x$ because $S = \{x\} \cup S_x$. Thus,

$$\begin{aligned} S' &= (S \setminus \{v\}) \cup \{u\} \\ &= [(\{x\} \cup S_x) \setminus \{v\}] \cup \{u\} \\ &= \big[(\{x\} \cup (S_x \setminus \{v\}))\big] \cup \{u\}. \end{aligned}$$

Since $\{x\}$ is a dominating set of G, it follows that S' is a dominating set of G. Accordingly, S is a secure dominating set of G. ■

Lemma 2.7 *Let $G = x + H$ where H is a nontrivial connected graph. If S is a secure dominating set of H where $S \neq V(H)$, then S is a secure dominating set of G.*

Proof **:** Suppose that S is a secure dominating set of H where $S \neq V(H)$. Then S is a dominating set of $G = x + H$. Let $u \in V(G) \setminus S$. Then there exists $v \in S$ such that $uv \in E(G)$. If $u = x$, then $S' = (S \setminus \{v\}) \cup \{x\}$. Since $\{x\}$ is a dominating set of G, S' is also a dominating set of G. If $u \neq x$, then $u \in V(H) \setminus S$. Since S is a secure dominating set of H, $S' = (S \setminus \{v\}) \cup \{u\}$ is a dominating set of H and hence a dominating set of $G = x + H$. Therefore, S is a secure dominating set of S. ■

The next result shows the characterization of a fair secure dominating set of a graph $G = x + H$.

Theorem 2.8 *Let $G = x + H$ where H is a nontrivial connected graph. Then a nonempty subset S of $V(G)$ is a fair secure dominating set if and only if one of the following is satisfied:*
(i) $S = \{x\}$ and H *is complete.*
(ii) $S = \{x\} \cup S_x$ where S_x is fair dominating set of H.
(iii) $S = S_x$ where S_x is a secure $|S_x|$-fair dominating set of H or $S_x = V(H)$.

Proof **:** Suppose that a nonempty subset S of $V(G)$ is a fair secure dominating set. Consider the following cases:

Case 1. Suppose that $|S| = 1$. Let $S = \{x\}$. In view of Remark 2.5, $G = x + H$ must be a complete graph. Hence H is complete. This proves statement (i).

Case 2. Suppose that $|S| \neq 1$. First, if $x \in S$, then let $S = \{x\} \cup S_x$ where S_x is a nonempty proper subset of H. If G is complete, then H is also complete, that is, $\gamma_{fsd}(H) = 1$ by Remark 2.5. Thus, $S_x = \{v\}$ is a fair dominating set of H. If G is non-complete, then H is also non-complete. Let $v \in S_x$. Suppose to the contrary S_x is not a fair dominating set of H. Since H is non-complete connected graph, $|V(H)| \geq 3$. Let $u, z \in V(H) \setminus S_v$ with $z \neq u$ such that $|N_H(u) \cap S_x| \neq |N_H(z) \cap S_x|$. Thus, for every $u, z \in V(G) \setminus S$,

$$\begin{aligned} |N_G(u) \cap S| &= |(N_H(u) \cup \{x\}) \cap S| \\ &= |(N_H(u) \cap S) \cup (\{x\} \cap S)| \\ &= |[N_H(u) \cap (\{x\} \cup S_x)] \cup \{x\}| \\ &= |[(N_H(u) \cap \{x\}) \cup (N_H(u) \cap S_x)] \cup \{x\}| \\ &= |[\emptyset \cup (N_H(u) \cap S_x)] \cup \{x\}| \\ &= |(N_H(u) \cap S_x) \cup \{x\}| \\ &\neq |(N_H(z) \cap S_x) \cup \{x\}| \\ &= |(N_H(z) \cup \{x\}) \cap (S_x \cup \{x\})| \\ &= |N_G(z) \cap S|. \end{aligned}$$

This implies that S is not a fair dominating set of G contrary to our assumption. Hence, S_x must be a fair dominating set of H. This proves statement (ii). Next, if $x \notin S$, then let $S = S_x$ where S_x is a nonempty subset of H. If $S_x = V(H)$, then we are done with the proof of statement (iii). Suppose that $S_x \neq V(H)$. Then S_x is a nonempty proper subset of H. Further, S_x is a fair dominating set of G (since $S = S_x$) and hence a fair dominating set of H. Suppose S_x is not an $|S_x|$-fair dominating set of H. Then $|N_H(u) \cap S_x| \neq |S_x|$. Clearly, $N_G(x) \cap S_x = S_x$. Let $u \in V(G) \setminus S$. Then

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$$\begin{aligned} |N_G(u) \cap S| &= |(N_H(u) \cup \{x\}) \cap S_x|, \text{ since } S = S_x \\ &= |(N_H(u) \cap S_x) \cup (\{x\} \cap S_x)| \\ &= |(N_H(u) \cap S_x) \cup \emptyset|, \text{ since } \{x\} \cap S_x = \emptyset \\ &= |N_H(u) \cap S_x| \\ &\neq |S_x| = |N_G(x) \cap S_x| = |N_G(x) \cap S|. \end{aligned}$$

Thus, $|N_G(u) \cap S| \neq |N_G(x) \cap S|$ where $u, x \in V(G) \setminus S$. This is contrary to our assumption that S is a fair dominating set of G. Hence, S_x must be an $|S_x|$-fair dominating set of H proving statement (iii).

For the converse, suppose that statement (i) is satisfied. Then $S = \{x\}$ and H is complete. This implies that $G = x + H$ is complete. By Remark 2.5, $S$is a fair secure dominating set of G.

Next, suppose that (ii) is satisfied. Then $S = \{x\} \cup S_x$ where S_x is fair dominating set of H. Since H is nontrivial connected graph, $|V(H)| \geq 2$. If $|V(H)| = 2$, then let $S_x = \{v\}$. Clearly, the set $S = \{x, v\}$ is a fair secure dominating set of G. Suppose that $V(H) \neq 2$. Let $u, z \in V(H) \setminus S_x$. Then $|N_H(u) \setminus S_x| = |N_H(z) \setminus S_x|$. Thus, for every $u, z \in V(G) \setminus S$,

$$\begin{aligned} |N_G(u) \cap S| &= |(N_H(u) \cup \{x\}) \cap S| \\ &= |(N_H(u) \cap S) \cup (\{x\} \cap S)| \\ &= |[N_H(u) \cap (\{x\} \cup S_x)] \cup \{x\}|, \\ &\quad \text{since } S = \{x\} \cup S_x \\ &= |[(N_H(u) \cap \{x\}) \cup (N_H(u) \cap S_x)] \cup \{x\}| \\ &= |[\emptyset \cup (N_H(u) \cap S_x)] \cup \{x\}| \\ &= |(N_H(u) \cap S_x) \cup \{x\}| \\ &= |(N_H(z) \cap S_x) \cup \{x\}| \\ &= |(N_H(z) \cup \{x\}) \cap (S_x \cup \{x\})| \\ &= |N_G(z) \cap S|. \end{aligned}$$

This implies that S is a fair dominating set of G. Since S_x is a dominating set of H, it follows that $S = \{x\} \cup S_x$ is a secure dominating set of G by Lemma 2.6. Accordingly, S is a fair secure dominating set of G.

Finally, suppose that (iii) is satisfied. Then $S = S_x$ where S_x is an $|S_x|$-fair secure dominating set of H^x or $S_x = V(H^x)$. Consider that $S_x = V(H^x)$. Clearly, $S = V(H)$ is a fair secure dominating set of $G = x + H$. Consider that S_x is an $|S_x|$-fair dominating set of H. Let $u \in V(H) \setminus S_x$. Then $|N_H(u) \cap S_x| = |S_x|$. Clearly, $N_G(x) \cap S_x = S_x$. Let $u \in V(G) \setminus S$. Then

$$\begin{aligned} |N_G(u) \cap S| &= |(N_H(u) \cup \{x\}) \cap S_x|, \text{ since } S = S_x \\ &= |(N_H(u) \cap S_x) \cup (\{x\} \cap S\_x)| \\ &= |(N_H(u) \cap S_x) \cup \emptyset|, \text{ since } \{x\} \cap S_x = \emptyset \\ &= |N_H(u) \cap S_x| \\ &= |S_x| \\ &= |N_G(x) \cap S_x| \\ &= |N_G(x) \cap S|. \end{aligned}$$

Thus, $|N_G(u) \cap S| = |N_G(x) \cap S|$ where $u, x \in V(G) \setminus S$. This means that S is a fair dominating set of G. Since S is a secure dominating set of H, it follows that S is a secure dominating set of $G = x + H$ by Lemma 2.7. Accordingly, S is a fair secure dominating set of G. ∎

We need the following definition and remark for the characterization of a fair secure dominating set in the corona of two graphs.

Definition 2.9 Let G and H be graphs of order m and n, respectively. The *corona* of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and m copies of H, and then joining the ith vertex of G to every vertex of the ith copy of H. The join of vertex v of G and a copy Hv of H in the corona of G and H is denoted by $v + H^v$.

Remark 2.10 *Let G and H be nontrivial connected graphs. Then $\gamma(G \circ H) = |V(H)|$.*

The following result is the characterization of the fair secure dominating set resulting from the corona of two graphs.

Theorem 2.11 *Let G and H be nontrivial connected graphs. Then a nonempty subset S of $V(G \circ H)$ is a fair secure dominating set if and only if one of the following is satisfied:*

(i) *$S = V(G)$ and H is complete.*

(ii) *$S = V(G) \cup \left(\bigcup_{v \in V(G) \setminus X} V(H^v)\right) \cup (\bigcup_{x \in X} S_x)$ where $X \subseteq V(G)$ and S_x is fair dominating set of H^x.*

(iii) *$S = \bigcup_{x \in V(G)} S_x$ where S_x is an $|S_x|$-fair secure dominating set of H^x or $S_x = V(H^x)$.*

Proof **:** Suppose that a nonempty subset S of $V(G \circ H)$ is a fair secure dominating set. Consider the following cases.

Case 1. Suppose that $V(G) \cap S \neq \emptyset$. By Remark 2.10 and Remark 2.4, $|V(G)| = \gamma(G \circ H) \leq \gamma_{fsd}(G \circ H) \leq |S|$. This means that $V(G) \subseteq S$.

First, consider that $S = V(G)$. Let $x \in S$. Since S is a fair secure dominating set of $G \circ H$, $\{x\}$ must be a fair secure dominating set of $x + H^x$. This implies that for each $x \in S$, H^x is complete by Theorem 2.8. Thus, $S = V(G)$ and H is complete, proving statement (i).

Next, consider that $S \neq V(G)$.Then $V(G) \subset S$. Let $S = V(G) \cup \left(\bigcup_{v \in V(G) \setminus X} V(H^v)\right) \cup (\bigcup_{x \in X} S_x)$, $X \subseteq V(G)$ and $S_x \subset V(H^x)$ for all $x \in X$. If $X = V(G)$, then $\bigcup_{v \in V(G) \setminus X} V(H^v) = \emptyset$. Thus $S = V(G) \cup \left(\bigcup_{x \in V(G)} S_x\right)$. Since H is nontrivial connected graph, $|V(H)| \geq 2$. Suppose that $|V(H)| = 2$. Clearly, S_x is a fair dominating set of H^x for all $x \in V(G)$. Suppose that $|V(H)| \neq 2$. Then $|V(H)| \geq 3$. Let $u, z \in V(H^x) \setminus S_x$ for all $x \in V(G)$.
Since S is a fair dominating set of $G \circ H$, $|N_{G \circ H}(u) \cap S| = |N_{G \circ H}(z) \cap S|$ for all $u, z \in V(G \circ H) \setminus S$. Thus, for all $x \in V(G)$

$$\begin{aligned} |(N_{H^x}(u) \cap S_x)| + |\{x\}| &= |(N_{H^x}(u) \cap S_x) \cup \{x\}| \\ &= |(N_{H^x}(u) \cup \{x\}) \cap (S_x \cup \{x\})| \\ &= |N_{G \circ H}(u) \cap S| \\ &= |N_{G \circ H}(z) \cap S| \end{aligned}$$

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$$= |(N_{H^x}(z) \cup \{x\}) \cap (S_x \cup \{x\})|$$
$$= |(N_{H^x}(z) \cap S_x) \cup \{x\}|$$
$$= |(N_{H^x}(z) \cap S_x)| + |\{x\}|.$$

This implies that $|(N_{H^x}(u) \cap S_x)| = |(N_{H^x}(u) \cap S_x)|$ for all $x \in V(G)$. Hence, S_x is a fair dominating set of H^x for all $x \in V(G)$. Thus, $S = V(G) \cup (\bigcup_{x\in X} S_x)$ where $X = V(G)$ and S_x is fair dominating set of H^x. Similarly, if $X \neq V(G)$, then $S = V(G) \cup \left(\bigcup_{v\in V(G)\setminus X} V(H^v)\right) \cup (\bigcup_{x\in X} S_x)$ *where* $X \subset V(G)$ *and* S_x *is fair dominating set of* H^x. This complete the proofs of statement (ii).

Case 2. Suppose that $V(G) \cap S = 0$. Let $S = \bigcup_{x\in V(G)} S_x$ *where* $S_x \subseteq V(H^x)$. If $S_x = V(H^x)$ for all $x \in V(G)$, then the proofs of statement (iii) is done. Suppose that $S_x \neq V(H^x)$ for all $x \in V(G)$. Then $S_x \subset V(H^x)$ for all $x \in V(G)$. Clearly, if $|V(H)| = 2$, then S_x is an $|S_x|$-fair secure dominating set of H^x. Suppose that $|V(H)| \neq 2$. Then $|V(H)| \geq 3$. Let $u, z \in V(H^x) \setminus S_x$ for all $x \in V(G)$. Since S is a fair dominating set of $G \circ H$, $|N_{G\circ H}(u) \cap S| = |N_{G\circ H}(z) \cap S|$ for all $u, z \in V(G \circ H) \setminus S$. Thus, for all $x \in V(G)$

$$|(N_{H^x}(u) \cap S_x)| = |N_{G\circ H}(u) \cap S|$$
$$= |N_{G\circ H}(z) \cap S|$$
$$= |N_{H^x}(z) \cap S_x|.$$

This implies that S_x is a fair dominating set of H^x for all $x \in V(G)$. Let $u \in V(G \circ H) \setminus S$. Since S is a secure dominating set of $G \circ H$ there exists $v \in S$ such that $uv \in E(G \circ H)$ and $S' = (S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G \circ H$. Now, $u \in V(H^x) \setminus S_x \subset V(G \circ H) \setminus S$, there exists $v \in S_x \subset S$ such that $uv \in E(H^x)$ and $S'_x = (S_x \setminus \{v\}) \cup \{u\}$ is dominating set of H^x for each $x \in V(H)$. This implies that S_x is a fair secure dominating set of H^x for all $x \in V(G)$. Further, for all $x \in V(G)$, Let $u \in V(H^x) \setminus S_x$. Then

$$|(N_{H^x}(u) \cap S_x)| = |N_{G\circ H}(u) \cap S|$$
$$= |N_{G\circ H}(x) \cap S|$$
$$= |N_{G\circ H}(x) \cap S_x|.$$
$$= |S_x|.$$

This implies that S_x is an S_x is an $|S_x|$-fair secure dominating set of H^x. This complete the proofs of statement (iii).

For the converse, suppose that statement (i) is satisfied. Then $S = V(G)$ and H is complete. Let $x \in V(G)$. Then $\{x\}$ is a fair secure dominating set of $x + H^x$ by Theorem 2.8 (i). Clearly, $\bigcup_{x\in V(G)} \{x\} = V(G)$ is a fair secure dominating set of $\langle \bigcup_{x\in V(G)} V(x + H^x)\rangle = G \circ H$. Thus, S is a fair secure dominating set of $G \circ H$.

Suppose that statement (ii) is satisfied. Then $S = V(G) \cup \left(\bigcup_{v\in V(G)\setminus X} V(H^v)\right) \cup (\bigcup_{x\in X} S_x)$ where $X \subseteq V(G)$ and S_x is fair dominating set of H^x. If $V(G) = X$, then $\{x\} \cup S_x$ is a fair secure dominating set of $x + H^x$ for all $x \in V(G)$ where S_x is fair dominating set of H^x, by Theorem 2.8(ii). Clearly, $\bigcup_{x\in V(G)} (\{x\} \cup S_x) = V(G) \cup \left(\bigcup_{x\in V(G)} S_x\right) = S$ is a fair secure dominating set of $\langle \bigcup_{x\in V(G)} V(x + H^x)\rangle = G \circ H$. Similarly, if $X \subset V(G)$, then $S = V(G) \cup \left(\bigcup_{v\in V(G)\setminus X} V(H^v)\right) \cup (\bigcup_{x\in X} S_x)$ is a fair secure dominating set of $G \circ H$.

Suppose that statement (iii) is satisfied. Then $S = \bigcup_{x\in V(G)} S_x$ where S_x is an $|S_x|$-fair secure dominating set of H^x or $S_x = V(H^x)$. If $S_x = V(H^x)$, then by Theorem 2.8(iii), $V(H^x)$ is a fair secure dominating set of $x + H^x$ for all $x \in V(G)$. Clearly, $\bigcup_{x\in V(G)} V(H^x) = S$ is a fair secure dominating set of $\langle \bigcup_{x\in V(G)} V(x + H^x)\rangle = G \circ H$. If S_x is an $|S_x|$-fair secure dominating set of H^x, then S_x is a fair secure dominating set of $x + H^x$ by Theorem 2.8 (iii). Clearly, $\bigcup_{x\in V(G)} S_x = S$ is a fair secure dominating set of $\langle \bigcup_{x\in V(G)} V(x + H^x)\rangle = G \circ H$. ∎

The following result is an immediate consequence of Theorem 2.11.

Corollary 2.12 *Let G and H be nontrivial connected graphs. Then*

$$\gamma_{fsd}(G \circ H) = \begin{cases} |V(G)| & \text{if } H \text{ is complete} \\ |V(G)||S_x| & \text{if } S_x \text{ is an } |S_x| - \text{fair secure dominating set of } H^x \ \forall x \in V(G) \end{cases}$$

Proof: If H is complete, then by Theorem 2.11(i), $V(G)$ is a fair secure dominating set of $G \circ H$. This implies that $\gamma_{fsd}(G \circ H) \leq |V(G)|$. In view of the Remark 2.10 and Remark 2.4, $|V(G)| = \gamma(G \circ H) \leq \gamma_{fsd}(G \circ H)$. This shows that $\gamma_{fsd}(G \circ H) = |V(G)|$.

If S_x is an $|S_x|$-fair secure dominating set of H^x for all $x \in V(G)$, then $S = \bigcup_{x\in V(G)} S_x$ is a fair secure dominating set of $G \circ H$ by Theorem 2.11(iii).

If H is complete, then by Remark 2.5, $S_x = \{v\}$ is a fair secure dominating set of H^x. Thus,

$$\gamma_{fsd}(G \circ H) \leq |S| = \left|\bigcup_{x\in V(G)} S_x\right|$$
$$= |V(G)||S_x| = |V(G)| \cdot 1.$$

Since $|V(G)| = \gamma(G \circ H) \leq \gamma_{fsd}(G \circ H)$, it follows that $\gamma_{fsd}(G \circ H) = |V(G)| \cdot 1 = |V(G)||S_x|$.

Suppose that H is non-complete. Then

$\gamma_{fsd}(G \circ H) \leq |S| = |V(G)||S_x|$ for all $S_x \subset V(H^x)$. Thus,

$$\gamma_{fsd}(G \circ H) \leq |V(G)||S_x|.$$

Suppose S^0 is a minimum fair secure dominating set of $G \circ H$. Then

$$\gamma_{fsd}(G \circ H) = |S^0| = \left|\bigcup_{x\in V(G)} S_x\right|$$
$$= |V(G)||S_x| \geq |V(G)||S'_x|.$$

Thus,

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$$\gamma_{fsd}(G \circ H) \geq |V(G)||S_x'|$$

where S_x' is a minimum $|S_x|$ -fair secure dominating set of H^x for all $x \in V(G)$. Accordingly,

$$\gamma_{fsd}(G \circ H) = |V(G)||S_x|. \blacksquare$$

III. CONCLUSION AND RECOMMENDATIONS

In this work, we extend the concept of the fair secure domination of graphs. The fair secure domination in the corona of two connected nontrivial graphs was characterized. The exact fair secure domination number resulting from the corona of two connected nontrivial graphs was computed. This study will guide us to new research such bounds and other binary operations of two connected graphs. Other parameters relating the fair secure domination in graphs may also be explored. Finally, the characterization of a fair secure domination in graphs and its bounds is a promising extension of this study.

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